Topic 4-
Functions

Mopic 4-Functions

We are going to formally define functions as sets but then
after that we won't really Topic 4-Functions)
We are going to formally define
functions as sets but then
after that we won't really
use that method anymore
use that method anymore
usual. use that method anymore we will just use formulas like usual .

Ex: Consider the function $f(x) = x^2$ where $x \in \mathbb{R}$.
 $f(x) = x^2$ where $x \in \mathbb{R}$.
 $f(x) = x^2$ where $x \in \mathbb{R}$. R jgraph of f $\begin{array}{ccc} \uparrow & & \downarrow \end{array}$ (-2) $\left\langle \begin{array}{c} \mathbf{0} \end{array} \right\rangle$ \downarrow $x: Consider the function
\n $\zeta(x) = x^2$ where $x \in \mathbb{R}$.
\n $(-2)^4$ \uparrow $(2, 4)$
\n $(-1, 1)$ $(-1, 1)$
\n $(0, 0)$
\n $(0, 0)$
\n $(0, 0)$
\n $(0, 0)$
\nR
\nR
\nQ
\nR
\nQ
\nR
\nR
\nQ
\nQ
\nR
\nR$ \bigvee The graph is $\big\{\;(\;\times,\;$ $x^{2})|x\in\mathbb{R}^{2}$ This graph lives inside of $\mathbb{R}\times\mathbb{R}$ P N V
Sinside of
Co-domain, where
the range lives RXIK
Co-domain, lives

 $EX: f(x,y) = x^{2} + y^{2}$ graph lives in $R^2 = R \times R \times \frac{IR}{A}$ Co-domain domair

R: Let ^A and ^B be sets . Let f be ^a subset of AXB. We say that function from ^A to ^B if this is saying that we ① for every a f ^A there can plug ^a into f exists be ^B where I to get b. & ie f(x) ⁼ b (a, b) Ef and vertical ^② if (a, b) Ef and 1 line

 $(a, b, c) \in f$ $+$ hen bi⁼b₂ $|$ test If this is the case then we LT This is $A \rightarrow B$ to mean that NILLE T. I'LL IC
F is a function from A to B

The set A is called the
domain of f.
The set B is called the
co-domain of f.
If (a,b)
$$
\in
$$
 f then
we write f(a) = b
The range of f is
range(f) = $\{b \in B |$ there exists a $\in A$

$$
E \times: A = \{-1, 100, 3, \frac{1}{3}\}
$$
\n
$$
B = \{T, -12, -1, \frac{1}{2}, 17, 14\}
$$
\n
$$
+ = \{(-1, -1), (100, \pi), (3, 17), (\frac{1}{3}, -1)\}
$$
\n
$$
+ (-1) = -1 + (100) = \pi + (13) = 17
$$
\n
$$
F(-1) = -1 + (100) = \pi + (13) = 17
$$
\n
$$
F(-1) = \frac{12}{100}
$$
\n
$$
F(-1) = \frac{12}{100}
$$
\n
$$
F(-1) = \frac{12}{14}
$$
\n
$$
F(-1) = \frac{12}{14}
$$

\n
$$
Is f a function from A + B?
$$
\n
\n $0 + is defined on all of A \oslash$ \n
\n $0 = 1$ is defined in more than one element of B\n

\n\n $1e_{s} + is a function from A + B.$ \n

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\n\n $1e_{s} + is a function from A + B.$ \n

$$
EX: A = \{1, 100, 3, \frac{1}{3}\}
$$

\n
$$
B = \{1, 100, 3, \frac{1}{3}\}
$$

\n
$$
B = \{(100, \pi), (3, 17), (\frac{1}{3}, -1), (100, -12)\}
$$

\n
$$
9 = \{(100, \pi), (3, 17), (\frac{1}{3}, -1), (100, -12)\}
$$

\n
$$
9^{(100)=\pi}, 9^{(3)=17}, 9^{(\frac{1}{3})=-1}, \frac{9^{(100)=-12}}{1, -12}
$$

\n
$$
9 = \{1, 1, 2, 3, 4, 5, 6, 7, 7\}
$$

\n
$$
100 = (-1) \text{ is not defined } \times
$$

 $\frac{1}{2}$ g (100) has two $\frac{1}{2}$ Values : TT & -12 ⁹ is not ^a function from ^A to B .

 $29 (100)$ has two
youves: π $k-12$
9 is not a function from A to B.
Let's now use furnulas
to define functions
instead of defining Let's now use formulas to define functions instead of defining them as subsets nstead of
hem as su
of AxB.

EX : Let A be any non-empty $\frac{2}{2}$ set . Let H be uny non
The identity function on A is the function $\stackrel{\circ}{\mathcal{L}}_{\Delta}$: $\stackrel{\circ}{\mathcal{A}} \longrightarrow \stackrel{\wedge}{\mathcal{A}}$ defined as $i_{A}(x) = x$ for all $x \in A$. Sometimes we will just write i instead of i_A. Formally you can think of Formally you can think or
 $I_{\mathcal{A}} = \{ (x, x) \mid x \in A \} \subseteq A x A$ Ex: Let A be any non-empty
set. The identity function
on A is the function
 λ_{A} : A \rightarrow A
defined as
 $\lambda_{A}(x) = x$ for all $x \in A$.
Sometimes we will just
write i instead of λ_{A}
Formally you can think of
Formally you $\hat{+}$ $\frac{y}{x}$
(x,x)
 $\frac{y}{x}$
 $\frac{y}{x}$ $\widetilde{\times}$

 $A = \{1, 2, 3, 4\}$ $\overline{\lambda}_{\mathsf{A}}(1)=1$ N_{A} $\bar{\Lambda}_{A}(2)=2$ $\bar{\lambda}_{A}(3)=3$ $\tilde{\lambda}_{A}(4)=4$ $A = \mathbb{R}$, $\overline{\lambda}_{\mathbb{R}}$, $\overline{\mathbb{R}}$, $\mathbb{R} \rightarrow \mathbb{R}$, $\overline{\lambda}_{\mathbb{R}}(x) = x$ EX graph $(2,1)$ way $O(1,1)$ μ (∂/∂) draw $M_{\mathbb{R}}$

Ex: Let nez, $M \geqslant$ 2 . map is Ex: Let nEZ, n>2.
Define the reduction $\overbrace{m \circ d}^{is}$ another module n map $\frac{x: Let neZ}{\frac{1}{x \cdot 0.000} + he}$
 $\frac{1}{x \cdot 0.000} = \frac{1}{x \cdot 0.000}$
 $\frac{1}{x \cdot 0.000} = \frac{1}{x \cdot 0.000}$ $T_{n}: \mathbb{Z} \longrightarrow \mathbb{Z}$ or T_{n} some that for
for Fraction Where $\pi_{n}(x) = x$ Ex: Let $n \in \mathbb{Z}$, $n \ge 2$. map

Define the reduction

modulo n map
 $\pi_n : \mathbb{Z} \longrightarrow \mathbb{Z}$

Where $\pi_n (x) = x$ mapping

Where $\pi_n (x) = x$ mapping
 $\frac{EX}{x}$ in = 3
 $\mathbb{Z}_3 = \{ 5, 7, 7 \}$
 $\pi_3 : \mathbb{Z} \rightarrow \mathbb{Z}_3$, $\pi_3 (x) = x$ $\pi_n: \mathbb{Z} \longrightarrow \mathbb{Z}$
 $\frac{1}{\pi \log n} \longrightarrow \mathbb{Z}$ $Ex: n = 3$ Ex: $\frac{21}{2}$ = $\frac{2}{2}$ $T, \overline{2}\}$ $\pi_{3}: \mathbb{Z} \rightarrow \mathbb{Z}_{3}$ \int
 $\pi_3(x) = \overline{x}$ some computations are :

$Some$ computations are
 $\Pi_3(0) = \overline{0}$ $\Pi_3(\Pi_3(-1) = -1 = 2$ $\pi_3(0) = 0$
 $\pi_3(1) = 1$ $\Pi_{3}(-2) = -2 = 1$

 $\Pi_3(2) = 2$ $\Pi_{3}(3)=\overline{3}=\overline{0}$ $\Pi_{3}(4)=\overline{4}=\overline{1}$ $\pi_3(5) = 5 = 2$

 $\pi_3(-3) = -3 = 0$ $\Pi_{3}(-4) = -4 = 2$ $\pi_{3}(-5)=-5=1$

$domain(\pi_{2}) = \mathbb{Z}$ $co-domain(\pi_{3})=\frac{N}{3}$ $\begin{aligned} \text{C} & \text{o} - \text{do} \text{maia}(\pi_3) = \\ \text{range} & (\pi_3) = \frac{5}{2} \pi_1 \end{aligned}$ T, \overline{z} } = \overline{z}

definedfunctions

Ex: Suppose you and your friend Francis want to define a function on Ch . You say "How about this y_1 x_2 . \cdots \vdots \vdots \vdots \Box \Box where $f(\frac{a}{b}) = \frac{b}{a}$ |
|}

Francis says "I don't know Francis says "I do
about that function. about that function. What about $f(\frac{0}{1}) = \frac{1}{0}$? That duesn't seem to make sense." about $f(\frac{0}{1})=\frac{1}{0}$? That
doesn't seem to make sense. duesn't seem to make sense.
You say "You're right. good call. $\left| \right|$ Then you say , "Ok I've got

an, the sidea. How a boot+
\ng:
$$
(A \rightarrow R)
$$
 where $g(\frac{a}{b}) = a$?
\nThat, both works. For example,
\n $g(\frac{3}{5}) = 3$ and $g(\frac{0}{2}) = 0$."
\nThen, Francis says, "Hey wait
\na minule, $g(\frac{3}{5}) = 3$ but $g(\frac{6}{10}) = 6$
\nand $\frac{3}{5} = \frac{6}{10}$. Show ldn't g
\nagree on those numbers?
\nYou say "Un year words"

Me functions fand y
above are not well-defined.

How to check that $f:A\rightarrow B$ is well-defined Check two things: Check two th
① If a E A , then f(a) EB ^② If some or all of the b e elements from A can expressed in more than one way then we must one way that if $a_{13}a_{2}$ are
check that if $a_{13}a_{2}$ are two expressions of the same I $+$ hen $+(a_{1}) = f(a_{2})$ How to check that $f: A \rightarrow B$

is well-defined

Check two things:

OIf $a \in A$, then $f(a) \in B$

2If some or all of the

elements from A can be

expressed in more than

one way then we must

check that if $a_{1,}a_{2}$ are

two e element in A (ie $a_1 = a_2$)

 $Ex: Let f: \mathbb{Q} \to \mathbb{Q}$ where $f(\frac{a}{b}) = (\frac{a}{b})^2.$ Is ^f well-defined ? Yes ^Y ।
० $f(\frac{\alpha}{b}) = (\frac{\alpha}{b})^2$.
 $Ts + well-defined? YesV$
 $proot that f is well-defined:$ $\frac{prod - det + is well-defined}{det - det}$ S o, a , $b \in \mathbb{Z}$ and $b \neq 0$. S_{0} , $a, b \in U$ where $\frac{a^{2}}{b^{2}}$
Then, $f(\frac{a}{b}) = (\frac{a}{b})^{2} = \frac{a^{2}}{b^{2}}$ Then, $f(\frac{a}{b}) = (\frac{a}{b}) = \frac{a}{b^2}$
We have that $a^2, b^2 \in \mathbb{Z}$ f well-defined? $\frac{100}{6}$ f that f is well-defined:
 $\frac{100}{6}$ f $\frac{100}{6}$ f $\frac{2}{6}$ and $\frac{100}{6}$
 $\frac{100}{6}$, $\frac{100}{6}$ = $\left(\frac{100}{6}\right)^2$ = $\frac{100}{6}$
 $\frac{100}{6}$ mave that $\frac{100}{6}$ b² = 2

and $\frac{$ and $b^2 \neq 0$ (since $b \neq 0$). So ,..
() ≤ 0
 $\in \mathbb{Q}$.

 (2) Suppose $\frac{\alpha}{b}$, $\frac{c}{d} \in \mathbb{C}$ and $\frac{\alpha}{b} = \frac{c}{d}$.

T5
$$
f(\frac{a}{b}) = f(\frac{c}{d})
$$
 B
\nMethod 1:
\nSince $\frac{c}{b} = \frac{c}{d}$, then by squaring
\nboth side we get $(\frac{a}{b})^2 = (\frac{c}{d})^2$
\nSo, $f(\frac{a}{b}) = f(\frac{c}{d})$
\n $\frac{y_{0v}}{b}$ might actually is this true?
\n
\nRecall how we define two fractions
\nthe equal:

&

Suppose
$$
\frac{a}{b} = \frac{c}{d}
$$
.
\nThen $ad = bc$.
\nSo, $(ad)^{2} = (bc)^{2}$
\n $\frac{1}{a} = \frac{1}{b}$
\nThen $ad^{2} = (bc)^{2}$
\n $\frac{1}{a} = \frac{1}{b}$
\nThen $ad^{2} = b^{2}$
\n $\frac{1}{a} = \frac{1}{d}$
\nSo, $\frac{a^{2}}{b^{2}} = \frac{c^{2}}{d^{2}}$
\nThus, $f(\frac{a}{b}) = f(\frac{c}{d})$

From 1 and 2 above f is well-defined.

Ex: Let nEZ, nz: $m \in \mathbb{Z}$. $Pick$ Define f: Zn > Ln $f_{\alpha}(\overline{x}) = \overline{\alpha} \cdot x$ \sqrt{v} do some examples $let's$ $n=4,$ $Z_{4}=\{5,7,2,3\}$ w hen U_{-} \mathbb{Z} 4 $f(\overline{0}) = 1.5 = 0$ $\Gamma = \overline{\Gamma} \cdot \overline{\Gamma} = \overline{\Gamma}$ $f(\overline{z})=\overline{1}\cdot\overline{2}=2$ \overline{z} $f(\overline{3})=\overline{1}\cdot\overline{3}=\overline{3}$ $\overline{}$

 $f'_{2}(\vec{0}) = \vec{2} \cdot \vec{0} = \vec{0}$ $f_{2}(\tau) = 2 \cdot \tau = 2$ $f_{2}(z) = 2.2 = 4$ \bigcirc $= 2.3 = 6$ $f_{2}(\tilde{3})$ Z

 $f_3(\overline{o}) = 3.0 = 0$ $f_{3}(\tau)=\overline{3}\cdot\overline{1}=\overline{3}$ $f_3(z) = 3.2z$ $6\overline{6}$ $\overline{2}$ \overline{q} $\sqrt{3}.3 =$ $f_{3}(\overline{3})$

 $f(x)$ $\overline{O} \cdot X$ heurem: Let nEZ, nz2. a E Z Let fai Zn+Zn $|e+$ given by $f_a(\overline{x}) = \overline{a} \cdot \overline{x}$. be Then ta is well-defined. proof: $X \in \mathbb{Z}$, where $x \in \mathbb{Z}$ $(I) \lfloor e \rfloor$

Since X, a EZ we Know axEZ. $Thu\zeta$ $f_{\alpha}(\overline{x}) = \overline{a} \cdot \overline{x} = \overline{ax} \in \mathbb{Z}_n$. $QLe+ \overline{x}, \overline{y} \in \mathbb{Z}$ where $\overline{x} = \overline{y}.$ Then, $(sine\overline{x}=\overline{y})$ $f_a(\overline{x}) = \overline{a} \cdot \overline{x} = \overline{a} \cdot \overline{y} = f_a(\overline{y}).$ when we talked about
well-defined operations we proved that $\begin{aligned} \begin{aligned} \hat{f} + \overline{b} &= \overline{c} \text{ and } \overline{d} = \overline{e} \end{aligned} \\ \begin{aligned} \text{then} \quad \overline{b} \cdot \overline{d} &= \overline{c} \cdot \overline{e} \end{aligned} \end{aligned}$

Def: Let A and B be sets. Let f: A > B be a function. We say that f is injective or one-to-une if the following is true: For all $a_{ij}a_{i} \in A_{j}$ if $a_1 \neq a_2$, then $f(a_1) \neq f(a_2)$ $f(a_1) = f(a_2)$ Ie $Y04$ cannot have $+M15$

$$
AnnHic way the time:\nFor all $a_{1,1}a_{2} \in A_{1}$
\n $if f(a_{1}) = f(a_{2}), then a_{1} = a_{2}$
$$

How to prove
$$
f: A\rightarrow B
$$
 is one-to-one
\nLet $a_{11}a_{2} \in A$.
\nSuples $f(a_{1}) = f(a_{2})$
\n $\therefore (p_{133}f \cdot shff)$
\n $\therefore (p_{133}f \cdot shff)$
\n $\therefore (p_{133}f \cdot shff)$

Ex: Let filR-IR be $\overline{de\$ frac{1}{2}} by $f(x) = -4x+5$ Let's prove ^f is one-to-one. Af: Suppose XxXzER and $f(x_1) = f(x_2)$. Then , $-4x_1+5=-4x_2+5.$ - S $Thvs, -4\times, = -4\times2$. So , X , $=$ X_2 . $f(x_2):$
 $= -4x_2+5.$
 $4x_2 \cdot 7 \xleftarrow[4]{x_2+5}$ $\frac{1}{4}$ Thus , f is one-to-one

How to show f: A > B is not $one-b-one$ $Find specific x₁, x₂ \in A$ where $x_1 \neq x_2$ $bvt + (\kappa_1) = f(x_2)$

Ex: Let ne Z, nz Z. Define f: Zn 7Zn by $f(\overline{x}) = (\overline{x})^2$

Claim: + is well-defined.

pf of claim:

where $x \in \mathbb{Z}$ (I) Given $\overline{X} \in \mathbb{Z}_n$ we have that $\zeta(\overline{\chi}) = \overline{\chi}^2 = \overline{\chi} \cdot \overline{\chi} = \chi^2.$ Since XEZ we Know $x^2 \in \mathbb{Z}$. Thus, $f(\overline{x}) = \overline{x^2} \in \mathbb{Z}$ 2) Suppuse X, XLE Zn and $\overline{X}_1 = \overline{X}_2$. Then, $f(\overline{\chi}_{1}) = \overline{\chi}_{1}^{2} = \overline{\chi}_{2}^{2} = f(\overline{\chi}_{2})$ $mvH:iswell-defined
inz_ny if $\overline{a}=\overline{c}$$ and $\overline{b}=\overline{d}$, then $\overline{D} = \overline{d} \overline{D}$ Use with $\overline{a} = \overline{b} = \overline{y}$ and $E = J = \overline{X}_{2}$ $ClGIM$

 $EX: n = 2, f(\overline{x}) = \overline{x}^2$

 $Clain: Lef f: Zn \rightarrow Zn$ Claim: Let $f: \mathbb{Z}_n \rightarrow \mathbb{Z}_n$
given by $f(\overline{x}) = \overline{x}^2$.
 \overline{x} $f(n > 2)$, then $f(\overline{x})$
 \overline{x} $f(n > 2)$, then $f(\overline{x})$
 \overline{x} $f(n > 2)$, then $f(\overline{x})$
Proof of claim:
Note first that since $n > 2$ given by $f(\overline{x}) = \overline{x}^2$. given by $f(x) - x$.
If $n > 2$, then f is not one-to-one . Proof of claim: Note first that since ⁿ > ² $+nni$
 $+nri$
 $T=T$ We know $Mhy \leq |Suppole T = T$. Then, $I = -1 (mod n)$.
Thus, $N \left(1 - (-1) \right)$ Note first that since n > 2

We know that $T \neq -1$

Why? Suppose $T = T$.

Then, $1 \equiv -1 \pmod{n}$.

Thus, $n \mid (1-(-1))$
 $T e$, $n \mid Z$.
 $T_n \cup S$, $n = \pm 1, \pm 2$.
 T_{max} , $n = \pm 1, \pm 2$. $Ie, nZ.$ $T_{n\vee5}$, $n=\pm1$ $±$ 2. 4 happen since $n > 2$

Ref: Let ^A and ^B be sets. Let $f: A \rightarrow B$. Let ^C be the range of f. + + · A > D.
Le the range of f.
that f is surjective We say or onto B if $C = B$. Let $f: A \rightarrow D$.

C be the range of f.

say that f is surjective
 $\frac{f}{f}$ $\frac{B}{f}$ $\frac{B}{f}$ $\frac{C}{f}$ $\frac{F}{f}$
 $\frac{B}{f}$ $\frac{F}{f}$ $\frac{C}{f}$ $\frac{F}{f}$ $\frac{F}{f}$
 $\frac{F}{f}$ $\frac{F}{f}$ $\frac{F}{f}$ $\frac{F}{f}$ $\frac{F}{f}$ $\frac{$

Another way to say: ^f is unto ^B if for every be B , o B IT ISTE aEA with $f(a) = b$.
Ex: Let f: R > R be defined $h\gamma + (x) = -4x+5.$ Let's show that f is onto R. $\mathcal{K}% _{0}\left(\mathcal{N}_{0}\right) =\mathcal{N}_{0}\left(\mathcal{N}_{0}\right)$ iK $P100f$: Let $b \in \mathbb{R}$. We must

find aeIR ScratchWork where $f(a) = b$. $f(a) = b$ Let $a=\frac{b-5}{-4}$. Note a E R and $-4a+5=6$
a= <u>b-5</u> $\begin{matrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{matrix}$ Find $a \in \mathbb{R}$ $\bigcup_{\text{where } f(a) = b} \bigg\{\n\begin{array}{l}\n\frac{Scrafdhwof}{f(a) = b} \\
\frac{bc-5}{f(a) = b} \\
\frac{bc-5}{f(a) = \frac{b-5}{-9}} \\
\frac{bc-5}{f(a) =$ $4(\frac{b-5}{-4})+5$ $=(b-5)+5=6.$ Thus , f is onto ^R. Find $a \in \mathbb{R}$

where $f(a) = b$. Scratch
 $e + a = \frac{b-5}{-4}$. $-4a$
 $\downarrow b = \frac{b-5}{-4}$. $-4a$
 $f(a) = f(\frac{b-5}{-4}) = -4(\frac{b-5}{-4})$
 $= (b-5)+5 = b$.

Thus, f is onto \mathbb{R} .

Thus, f is onto \mathbb{R} .

Thus, f is onto \mathbb{R} .

F $\sqrt{\frac{1}{3}}$ How to show $f: A \rightarrow B$ is not unto Find some be ^B where is ne $a \in A$ with $f(a) = b$ -= b

f is not unto: $proch: Let b = 2.$ f is n
proof:
Then. Then , $b \in \mathbb{N} \cup \{0\}$. roof: Let $b = C$.
Then, be NUS_0 .
But is no ae Z with $But is no aez with
+ (a)=2.$ Why ? If so, $Hnen$ $a = 2$. Then)
ر
(M $e + b = 2$
 $a \in \mathbb{N} \cup \{0\}$
 $a \in \mathbb{Z}$ with
 $a = \pm \sqrt{2} \notin \mathbb{Z}$
 $a = \pm \sqrt{2} \notin \mathbb{Z}$
 $a = \pm \sqrt{2} \notin \mathbb{Z}$ Thus , f is not onto n_{1} a = \pm (2 \neq
 n_{1} , \neq is not onto
because 2 \notin range $\frac{1}{16}$ not onto:

Then, be NU ? of.

But is no ae Z with

f(a) = 2.

Why?

If so, then $\alpha^2 = 2$.

Then, $\alpha = \pm \sqrt{2} \notin Z$. (f) .

Def: Let A and B be sets and f: A > B. We say that f is a bijection if It is one-to-one and onto B.

unto B f is not a bijection

Ex: (from Humework) Given $a \in \mathbb{Z}$, define $g_{a}: \mathbb{Z}_{n} \to \mathbb{Z}_{n}$ by $g_{a}(\overline{x}) = \overline{x} + \overline{a}$ $EX: 95: Z6 \rightarrow Z6, 95(\bar{x}) = \bar{x}+\bar{5}$ 16 \mathbb{Z}_{6} $95(2)=\overline{512}=7=1$ $95(5) = 0 + 5 = 5$ $95(1) = 1 + 5 = 6 = 0$

In the HW you show ga is Well-defined.

 fhe $CLain: Given a \in \mathbb{Z},$ function ya: Il n 7 Il giuen by $g_{\alpha}(\overline{x}) = \overline{x} + \overline{\alpha}$ is a bijection

 $P100f$

(one-to-one)
\nSuppose
$$
g_a(\overline{x}_1) = g_a(\overline{x}_2)
$$
 where
\n $\overline{x}_1, \overline{x}_2 \in \mathbb{Z}$ are
\nThen, $\overline{x}_1 + \overline{a} = \overline{x}_2 + \overline{a}$.
\nThen, $|\overline{x}_1 + \overline{a}| + \overline{-a} = |\overline{x}_2 + \overline{a}| + \overline{-a}$.
\nThus, $\overline{x}_1 + \overline{a} = \overline{x}_2 + \overline{a}$.
\nSo, $\overline{x}_1 = \overline{x}_2$.

Thus, ga is une-to-one. \mathbb{Z}_n U_{n} $(\sigma h \wedge \sigma)$ g_{α} Let JELLn, where y E Z. $y - \alpha$ $= y + -a$ Then, $y-a\in\mathbb{Z}$ n because y-uELL. $And,$ $D_{n}(\overline{y-a}) = \overline{y-a} + \overline{a}$ $= 4 - 0 + 0$ $\mu =$ onto. \sum O \int $9a$ 15

Def: Let A, B, C be sets. U et: Let A, D, C be sers.
Let $f: A \rightarrow B$ and $g: B \rightarrow C$. Define the <u>composition</u> م
پ B, C be
3 and 9:
Composition ^f and ^g to be the function (gof): A -> C by $(g \circ f)(a) = g(f(a))$ A B C a $f(f(a))$ 9 $g(f(a))$ $\begin{picture}(120,140) \put(0,0){\line(1,0){155}} \put(15,0){\line(1,0){155}} \put(15,0){\line(1,0){155}} \put(15,0){\line(1,0){155}} \put(15,0){\line(1,0){155}} \put(15,0){\line(1,0){155}} \put(15,0){\line(1,0){155}} \put(15,0){\line(1,0){155}} \put(15,0){\line(1,0){155}} \put(15,0){\line(1,0){155}} \put(15,0){\line(1,0){155$ Def: Let A, B, C be set

let f: A > B and $g: B$

letine the <u>composition</u>

i and g to be the

function (gof): A > 1

by (gof)(a) = g(f(a))

A B C

A B C

A B C

B C

B C

D g(! $\left(\begin{matrix} f(\alpha) \\ 0 \end{matrix}\right) = \left(\begin{matrix} 0 \\ 0 \\ 0 \end{matrix}\right) f(\alpha)$ $90 +$

 $EX: (Hannmock 12.4#9)$ Define f: $21 \times 21 \rightarrow 21$ where $f(m,n)=m+n$ and g: $\mathbb{Z} \rightarrow \mathbb{Z} \times \mathbb{Z}$ where $y(x) = (x^1)x^1$

 $F \circ g : \mathbb{Z} \to \mathbb{Z}$ $(f \circ g)(x) = f(g(x))$ $=\int (x,x) = x+x = 2x$ $\int u(x)$ $(f \circ g)(x) = 2x$ $Z\times L$ ZXZ $\sqrt{2}$ $(m+n,m+n)$ $m+n$ (m,n) $90f$ $90f: \mathbb{Z} \times \mathbb{Z} \rightarrow \mathbb{Z} \times \mathbb{Z}$ $(g \circ f)(m,n) = g(f(m,n))$

$$
=g(m+n)=(m+n, m+n)So, (g of)(m,n)=(m+n, m+n)
$$

 $Question: Is g l-l?$ Is gonto? $\leq |aim: gis| - 1$ $pf: Suppose g(x_1) = g(x_2)$ $9(x)$ Where $x_1, x_2 \in \mathbb{Z}$. Then, $(x_{1},x_{1})=(x_{2},x_{2})$ S_{o} , $X_{1} = X_{2}$.

Claim: g is not onto. $pf: Lef(1,2) \in \mathbb{Z} \times \mathbb{Z}$. There is no XEZ with $g(x) = (x, x) = (1, 2).$ $Z\times Z$ $\overline{\mathscr{L}}$ (x, x) \bigtimes $\int f(t) dt$ S_{\sim} $(1,2)$ \notin $nnge(g)$ $So, g is not one.$

Kecall that $f: \mathbb{Z} \times \mathbb{Z} \longrightarrow \mathbb{Z}$ where $f(m,n) = m+n$. Question: Is f onto? Is f 1-1?

Claim: f is onto ZxZ proof: Let yEZ. $(0, 9)$ Then, $(o,y) \in \mathbb{Z} \times \mathbb{Z}$ and $f(o,y) = o+y$ $(3, 2)$ $= 4$. $(5,0)$

Claimif is not 1-1 $\frac{\sum \{aim: f is not 1-1}{p(sof: f(3,2)=5=f(5,0)}\} }{b\cup f(3,2) \neq (5,0)},$
See picture above. (A) $\frac{\Gamma(\alpha_1,\alpha_2)}{\Gamma(\alpha_1,\alpha_2)} + (3,2) = 5 = 6$ 0) $\overline{b}v + (3,2) \neq (5,0).$ $\frac{1}{60!}$ (3,2) = (5,0)
See picture above. $\begin{array}{|l|l|l|}\hline \text{Claim: } f & \text{is not } l-1 \\
\hline \text{proof: } f(3,2)=5=f(5,0) \\
\hline \text{but } (3,2)\neq(5,0), \\
\hline \text{See prime above.} & \text{min}\n\end{array}$ Claim: f is not 1-1

Proof: f (3,2) = 5 = f (5,0)

but (3,2) \neq (5,0).

See picture above. (2)

Acovem: Let A, B, C be sets

and f : A \rightarrow B and g : B \rightarrow C. 11

Therem: Let ^A , B, ^C be sets and $f: A \rightarrow B$ and g $g: B \rightarrow C.$ ^④ If f and ^g are then got is onto $2IFF and g are hfh 1-l,$ then gof is 1-1.

\n
$$
\begin{array}{ccc}\n 3 & \text{if } f \text{ and } g \text{ are both} \\
 6 & \text{bigcfions} \\
 1 & \text{and only} \\
 4 & \text{then } g \text{ of } is a \text{bigcfin.} \\
 \end{array}
$$
\n

Then
$$
9^{\circ}f
$$
 is a bijection.

\nproof:

\nObserve f and g are both onto.

\nNote $9^{\circ}f$: $A \rightarrow C$.

Let $z \in C$. Since g is onto ^C \mathcal{C} there exists $y\in B$ where $g(y)=z$. there exists Since g is onto
 $y \in B$ where g
Since f is onto B y
Since f is onto B y $X\in A$ where $f(x)=y$. Then $(g\circ f)(x) = g(f(x))$ $= 9(y) = 2.$ S o, gof is unto Lbecause there exists XEA with $(90f)(x)=7$ (2) Suppose f and g are b . $b + b$ $| - |$.

 $Suppose (gef)(x_1) = (gef)(x_2)$ where x , X 2 E A . Then where x_{ij}
n, g (f(x_i $x_{2} \in A.$
 $\left(\int f(x_{2})\right).$ Since g is 1-1 and $g(f(x_1)) = g(f(x_2))$ $+hi\varsigma$ implies that $f(x_1) = f(x_2)$. Sincef is ^H and $f(x_1) = f(x_2) + his$ implies that $X_1 = X_2$. \int 0 $\int_{Q} (g \circ f)(x_1) = (g \circ f)(x_2).$ implies that X_{1} $=$ \times \overline{z} . Thus, gof is $|-|$.

^③ Supposef and^g are suppose 7 min (1-1 and onto).
both bijections (1-1 and onto). β y (, this implies that got will be onto . By ² , this implies that got will be 1-1. $\zeta \circ$ got is 3) Suppose f and 9
both bijections (1-1
By 1, this implies the
gof will be
gof will be
So, gof is a bi.
So, gof is a bi. $be \mid - \mid.$
a bijection.

-

So, got is a wist

Vow we talk about

inverse functions. talk about inverse functions.

 $(f^{-1} \circ f)(1) = f^{-1}(f(1)) = f^{-1}(1) = 1$ $(f^{-1}of)(z)=f^{-1}(f(z))=f^{-1}(1729)=Z$ $(f^{-1}{}_{0}f)(3) = f^{-1}(f(3)) = f^{-1}(TU) = 3$ Thus, $f^{\prime} \circ f = \lambda_{A}$ (the identity)

We see that see that
 $(f \circ f^{-1})(z) = z = i$ \tilde{L}_c (Z) f'' We see that
 $(f \circ f^{-1})(z) = z = \lambda_c(z)$

for all zec.

Def: Let A and B be sets

Def: Let A and B be sets

Let $f: A \rightarrow B$ be a one-to-one

Let $f: A \rightarrow B$ be a one-to-one

function. Let $C = \text{range}(f)$.

Define the inverse function. all $Z\in C$. $(f \circ f^{-1})(z) = z = \lambda_c(z)$
for all ZEC.
 $f \circ f^{-1} (z) = z = \lambda_c(z)$ Def: Let A and B be sets Let $f: A\rightarrow B$ be a $one-b-one$ Let $f: A \rightarrow B$ be a $C = range(f)$.
function. Let $C = range(f)$. function We see that
 $(f \circ f^{-1})(z) = z = \lambda_c(z)$

For all ZEC.
 $2ef: Let A and B be
\n $Qef: Let A and B be
\n $f \circ f^{-1} : A \rightarrow B be a one$
\n $Qef \circ f^{-1} : C \rightarrow Qef \circ f^{-1} : C \rightarrow Q$$$ inverse $C = range(f)$

rse function
 $f^{-1}: C \rightarrow A$ of f to be $f^{-1}: C \longrightarrow A$ such $f h a f^{-(z)}(z) = x$ where $f(x) = z$.

 β \bigcirc $f - 1$

Note: f⁻¹ is well-defined because f is one-to-one. There is one and only one arrow to reverse for each Zin C.

Theorem: Let A, B be sets.
\nLet f: A \to B be a one-to-one
\nfunction. Let C = range(f).
\nLet f': C \to A be the inverse
\nof f. Then:
\n
$$
\left(\begin{array}{c}\n\end{array}\right)
$$

\n $f \left(\begin{array}{c}\n\end{array}\right)$

(i) domain
$$
(f^{-1}) = range(f) = C
$$

\n(2) range $(f^{-1}) = domain(f) = A$

\nIn particular, f^{-1} is onto A.

\n(3) f^{-1} is one-to-one

\n(4) $(f^{-1} \circ f)(a) = a$ for all a $\in A$.

\n(50, $f^{-1} \circ f = \lambda_A$

Prof: ^① By det of f "we have d omain $(f^{-1}) = C =$ range (f) . domain (1)

(2) Let's show that range $(f^{-1}) = A$. By det of f⁻¹ (f) $c \nightharpoonup^{\texttt{f}}$ a y def of f^{-1}
we know
range (f-') \subseteq A.
 $\left(\begin{array}{c} 2 \\ \vdots \\ \vdots \\ 2\end{array}\right)$ $\frac{1}{\sqrt{2}}\int_{-\infty}^{\infty} f(x) \, dx$ $re^{-n\pi}$ \in A. $\begin{matrix}\n\text{angle}(t) \\
\text{angle}(f^{-1}) \\
\text{angle}(f^{-1}) \\
\text{angle}(f^{-1})\n\end{matrix}$ $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 &$ Why is $A \subseteq range(f^{-1})$. Let $a \in A$. Let $c = f(a)$ And, $f = f(a)$
= $f(a)$
 $f^{-1}(c) = a$ by def of f^{-1} ζ o, $+$ (c) - n "J
a Erange (f'). $Thvs,$ $f \in range(T)$
A $\subseteq range(F^{-1})$ Therefore, H = range (f").

 (3) Let's show that f^{-1} is $one-to-one$. $Suppose f^{-(}c_1) = f^{-(}c_2)$ Where c)
)
($\in C$. Where $c_1, c_2 \in C$.
We need to show that $c_1 = c_2$. We need to show that $c_1 =$
Let $\alpha = f^{-1}(c_1) = f^{-1}(c_2)$. $Sine$ $\alpha = f^{-1}(c_1)$ we know $f_{\text{in}}f(\alpha) = c_1$. $Since \quad a = f^{-1}(c_2)$ we know $+haf(f(a))=C_2.$ S_{0} , $C_{1} = f(a) =$ $\mathsf{C}_\mathcal{Z}$. S_{0} , $C_{1} = f(\alpha) = C_{2}$.
Thus, f⁻¹ is one-to-one.

$$
(4) Let's show that f^{-1} of = \bar{x}_{A}.
$$
\nLet a \in A.
\nSet c = f(a).
\nSo, f^{-1}(c) = a by def of f:
\nThen,
\n
$$
(f^{-1} \circ f)(a) = f^{-1}(f(a))
$$
\n
$$
= f^{-1}(c)
$$
\n
$$
= \alpha
$$
\n
$$
= \bar{x}_{A}(a)
$$
\nThus, $(f^{-1} \circ f)(a) = \bar{x}_{A}(a)$
\nfor all a \in A.
\nSo, f^{-1} \circ f = \bar{x}_{A}

⑤ Let's show that $(f \circ f^{-1})(c) = c$ for all ce^C . Let ceC . Let $c \in C$.
Then, $f^{-1}(c) = a$ where $a \in A$ and $f(a) = c$.

 $\bigcap_{n\in\mathbb{N}}\bigcup_{n\in\mathbb{N}}$ $(f \circ f^{-1})(c) = f(f^{-1}(c))$ $= f(\alpha)$

 $=$ \subset $= \bar{\lambda}_{C}(c)$

 $G \cup e + 9: C \rightarrow A$ where gof=ig We want to show that $g=f^{-1}$ So we must show that $g(c) = f^{-1}(c)$ for all $c \in C$. Let $c \in C$. Then, $f^{-1}(c) = a$ where $a \in A$ and $f(a) = c$. Ihen, a
n, $g(c) = g(f(a)) = (g \circ f)(a)$ $c \in C.$
 $f^{-1}(c) = a$
 $= A$ and $f = 0$
 $= g(f(a)) = \frac{g(f(a))}{g^o f = ap} = \frac{1}{g}$ $\stackrel{\bigstar}{=} \stackrel{\prime}{\mathcal{L}}_{A}(\alpha)$ $gof=IP$ = Q ..
= IA $= f^{-1}(c)$ Thus $, 9 = f^{-1}.$ $\langle\langle\langle\rangle\rangle\rangle$

 $f(4,5) = (4+5, 4+2.5) = (9,14)$ T (1) -
f (- 2) 1) = (-2+1) - $2 + 2$ $| \rceil = (-1, 0)$

Claim: f is one-to-one

Proof:

Suppose $f(m_1, n_1) = f(m_2, n_2)$

where (m_1, n_1) , $(m_2, n_2) \in \mathbb{Z}$

We need to show that (m_1, n_1) =

Since $f(m_1, n_1) = f(m_2, n_2)$ we $Lian: f is one-to-one$
proof:
 $S=60005e + (m_1, n_1) = f(m_2, n_2)$ $: f \circ \circ \circ f$: **I** $Supppose f(m_{1},n_{1}) = f(m_{2},n_{2})$ where $(m_{ij}n_{i})_{j}$ (M_{2J} $\left(M_{2},n_{2}\right)$
 $n_{2}\right) \in \mathbb{Z}\times\mathbb{Z}.$ where $(m_{ij}n_{1})$, $(m_{2j}n_{2})\in$
We need to show that $(m_{ij}$ N_1) = (M_2, M_2) . Since $f(m_{ij})$ n_{1}) = $f(m_{z1}n_{z})$ we know $fhat(M)$ $+$ n_{1} , m_{1} $+2n_1$ = (m_2+n_2) m_2 $\begin{array}{c} \n\therefore \text{ is one-to-on} \\ \n\therefore (m_{12}, n_{13}) = f(m_{21}, n_{23}) \\ \n\therefore n_{13} = f(m_{21}, n_{23}) \\ \n\therefore n_{13} = f(m_{21}, n_{23}) \\ \n\therefore n_{13} = m_{13} + n_{23} \n\end{array}$ $+ 2n_{2}$ $\overline{n e - \overline{n} - \overline{o} n e}$

(m₂, n₂) $\in \mathbb{Z} \times \mathbb{Z}$

(m₂, n₂) $\in \mathbb{Z} \times \mathbb{Z}$

+hat $(m_{ij}n_{i}) = (m_{2}n_{i})$

(m₂, n₂) we kno

(m₂, n₂) we kno

(m₂+n₂) m₂+2

(m₂+n₂) (m₂+n₂) m₂+2 We need to show that (m_1n_2)

Since $f(m_1n_1) = f(m_2n_2)$ is

that $(m_1+n_1,m_1+2n_1) = (m_2+n_2)$

Thus,
 $m_1+n_1 = m_2+n_2$ (1)
 $m_1+2n_1 = m_2+2n_2$ (2)

 $m_1 + n_1 = m_2 + n_2$
 $m_1 + 2n_1 = m_2 + 2n_2$ (2)

Calculating (2) - () we get

\nthat
$$
n_1 = n_2
$$
.

\nThus we get

\n
$$
m_1 + n_2 = m_1 + n_1 = m_2 + n_2
$$
\n
$$
\boxed{n_2 = n_1}
$$
\n
$$
\boxed{0_2 = n_1}
$$
\nSubtract n_2 from both sides to get $M_1 = m_2$.

\nThus, $(m_1, n_1) = (m_2, n_2)$.

\nThus, f is 0 are $-b$ - one.

\nClaim I)

 $\begin{array}{|l|l|} \hline \text{Claim 2: } f \text{ is one} \\ \hline \hline \text{Let (a,b) & \in \mathbb{Z} \times \mathbb{Z} \\ \text{We must find } (m,n) \in \\ \text{where } f(m,n) = (a,b) \\ \hline \text{where } f(m,n) = (a,b) \\ \hline \text{X} \times \text{Z} & \text{Z} \times \text{Z} \\ \hline \text{X} \times \text{Z} & \text{Z} \times \text{Z} \\ \hline \text{X} \times \text{Z} & \text{Z} \times \text{Z} \\ \hline \text{X} \times \text{Z} & \text{Z} \times \$ Claim Z: f is onto Let (a, $b) \in \mathbb{Z} \times \mathbb{Z}$. We must find (M, (n) $\in \mathbb{Z} \times \mathbb{Z}$ where f(m, $n) = (a, b)$ $H(m,n) = (a, k)$ (m,n) \rightarrow (m,n) $\begin{pmatrix} (m,n) & f & (a,b) \\ 0 & 0 & 0 \end{pmatrix}$ $\boldsymbol{\varrho}$ That is, we need to solve $(m+n, m+2n)$ is, we ne
(m+n, m+2n)
F(m, n) $=(\begin{smallmatrix} 0 & \ 0 & \end{smallmatrix})$. $\mathcal{F}(w'v)$ So we need to solve

$$
m+n = a
$$
\n
$$
m+2n = b
$$
\n
$$
m+n = a
$$
\n
$$
m+n = b-a
$$
\n
$$
m+n = b-a
$$
\nThen,\n
$$
m = a-n = a-(b-a) = 2a-b
$$
\n
$$
m = a-n = a-(b-a) = 2a-b
$$
\n
$$
m = a-n = a-(b-a) = 2a-b
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m = a-n = a-(b-a) = 2a-b
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m = a-n = a-(b-a) = 2a-b
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m = a-n = a-(b-a) = 2a-b
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m = a-n = a-(b-a) = 2a-b
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m = a-n = a-(b-a) = 2a-b
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m = a-n = a-(b-a) = 2a-b
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\n
$$
m = a-n = a-(b-a) = 2a-b
$$
\n
$$
m = a-n = a-(b-a) = 2a-b
$$

 $=(2a-b+b-a, 2a-b+2(b-a))$ $= (a, b)$

Thus, f is unto.

From above we have that f is 1-1 . Thus, e that
 $f^{-1} e x i s$ ts. And
domain $(f^{-1}) = range(f) = \mathbb{Z} \times \mathbb{Z}$ $(fisinh)$

 $Clain3:Le+ 9: ZxZ \rightarrow ZxZ$ From above we have that

f is 1-1. Thus, f⁻¹ exists.

And

domain (f⁻¹) = range (f) = $\mathbb{Z} \times \mathbb{Z}$

[f is onto]

(Claim 3: Let 9: $\mathbb{Z} \times \mathbb{Z} \rightarrow \mathbb{Z} \times \mathbb{Z}$

be defined by

9(a,b) = $(2a-b,b-a)$.

Then, $9 = f^{-1}$ be defined by g(a, $b_{0} = (2a-b,b-a).$ Then From above we have that
f is 1-1. Thus, f⁻¹ exists.
And
domain (f⁻¹) = range (f) = $\mathbb{Z} \times \mathbb{Z}$
 $\frac{1}{\sqrt{2 \times 2}}$ $(y)=2a-b$
n g = f -1 Let's use thm from last time : $f: A \rightarrow B$, f is $1-l$, $C= range(f)$

If
$$
g:C\rightarrow A
$$
 and $g\circ f = \bar{x}_{A}$
\nthen $g = f^{-1}$
\nProof of claim 3: We have
\n $(g \circ f)(m,n) = g(f(m,n))$
\n $= g(m+n, m+2n)$
\n $= (2(m+n)-(m+2n)g(m+2n)-(m+n))$
\n $= (m,n)$
\n $\therefore \bar{x}_{Z\times Z}(m,n)$.
\nSince $g \circ f = \bar{x}_{Z\times Z}$ we
\nhave $g = f^{-1}$. (Claim 3)

Def: Let A and B be Sets. Let f: A > B. O Let X = A The <u>image of X under</u> f is $f(\Sigma) = \frac{1}{2}f(x) | x \in \Sigma^2$ (2) Let $\overline{Y} \subseteq B$. The <u>inverse</u> image of I under f

 is the set $f^{-1}(\mathcal{I}) = \{a \in A \mid f(a) \in \mathcal{I}\}$ Note: We use f⁻¹ notation, but it duesn't necessarily mean inverse
function because for might not
tunction because for might paist

the following tunction. Consider LX: $f(1) = 7$ $f(z) = |z|$ $+(3)=7$ $f(y)=13$ \overline{O} $+(5) = 11$ \cdot | 2 $f(6)=12$ 9013 $[e + \overline{X} = \{ 2, 3, 5, 6 \}]$ Then, $f(\overline{X}) = \{f(2), f(3), f(5), f(6)\}$

 (b) Let $\bar{X} = \{1, 3, -5\}$ 10, 02 ر Then, $\pi_{G}(\overline{\chi}) = \left\{ \pi_{G}(1), \pi_{G}(3), \pi_{G}(-5), \pi_{G}(10), \right.$ $\pi_{6}(\log)\}$ $=\{7, 3, -5, 10, 102\}$ $=$ $\{$ $\bar{1}$, $\bar{3}$, $\bar{1}$, $\frac{1}{\sqrt{102}}$
 $\frac{1}{\sqrt{102}}$
 $\frac{1}{\sqrt{102}}$
 $\frac{1}{\sqrt{102}}$ - 42 $=\left\{\frac{1}{0},\frac{1}{1},\frac{1}{3},\frac{1}{4}\right\}$ -42

 (c) Let \overline{Y} = Σ T $\frac{C}{2}$. Let's calculate $\pi_{6}^{-1}(\Psi).$ Let's take a look at the picture.

 $Note: -$ 5, 1 N ote: , $7\in\overline{\mathcal{H}_{6}}^{1}(\mathcal{I})$

And , $-5 = 6(-1) + 1$ $1 = 6(0) + 1$ $7 = 6(1) + 1$ $A|$ sa \circ 135.77^{16} (I) and $13 = 6(2) + 1$. $Clain: T_6'(\Psi) = 26k+1|ke\mathbb{Z}$ Proof: (\subseteq) : Let $x \in \Pi_6^{-1}(\mathcal{I}).$

 $\frac{1}{\sqrt{2}}$ T hus, $\pi_{b}(x) = T$. Z
 $(X, T6)$
 $(X, T6)$
 $(Y, T6)$
 $(Y, T6)$
 $(Y, T6)$
 $(Y, T6)$ \int 0 $\pi_{6}(x)\in\overline{Y}$ S_0 , $\pi_6(x) \in \underline{Y}$

Thus, $\pi_6(x) = T$.
 S_0 , $\overline{x} = T$ in \mathbb{Z}_6

Then, $x \equiv 1 \pmod{6}$. Then $X = 1$ m 26
 $X = 1 \pmod{6}$. $Thus, 6|(x-1).$ Hence , $X-1 = 61$ where $1 \in \mathbb{Z}$. Therefore $\begin{matrix} \lambda \ \epsilon \end{matrix}$ $x = 61 + 1$. Thus, $x \in \{6k+1|keZ\}$ Hence, $\pi_{6}^{-1}(\Psi)\subseteq\{6k+1|k\in\mathbb{Z}\}$ (2) : Let y \in {6k+1 | keZ}

where le Z. $So, y = 6lt + 1$ Then, $\pi_{6}(y) = \overline{y} = 62 + 1$ $= 62 + 1$ $= 52+7$ $S_{\varphi}, \pi_{\varphi}(\varphi) \in \mathfrak{T}.$ \mathbb{Z}_6 $Thus, y \in \overline{\pi}_{6}^{1}(\mathcal{I}).$ Therefore, \S GR+1 | REZ $\Sigma = \overline{\pi_{6}}(\overline{x})$ $By (S) and (2), $\pi_G^{-1}(\mathcal{I}) = \{6k+1 | k \in \mathbb{Z}\}$$

Then:

() $f(wvz) = f(w)vf(z)$ Hw $\textcircled{2} f(w \cap Z) \subseteq f(w) \cap f(Z)$ Hammuck ^③ Give an example to show I #7, $8\,$ $+hat + (wnz) = f(w)nf(z)$ is not always true $(\forall) \text{If } W \subseteq \mathcal{Z}, \text{ then } f(w) \subseteq f(\mathcal{Z})$

proof: Let's prove (2), (3), then 1) $1 + h\epsilon_{n}(\mathfrak{F})$ (2) We want to show that $f(w\cap Z)\subseteq f(w)\cap f(Z)$. Let bEf(WNZ). $\left\{ \right\}$ W $f(wnz)$ WNZ Then there exists a EWNZ Where $f(\alpha) = b$.

Since a EWN² we know
\n
$$
a \in W
$$
 and $a \in Z$.
\nSince a E W and $f(a)=b$
\nwe know be $f(W)$
\nSince a E Z and $f(a)=b$
\nwe know be $f(Z)$.
\nThus, be $f(W) \cap f(Z)$.
\nHence, $f(W \cap Z) \subseteq f(W) \cap f(Z)$.
\n
\n3) Let's give an example
\nto show that
\n $f(W \cap Z) = f(W) \cap f(Z)$
\nis not always true.

$$
\overline{3} \text{Left's give an example} \n\begin{aligned}\n&\text{show that} \\
&\text{flow that} \\
&\text{flow } \uparrow \text{hat} \\
&\text{flow } \uparrow \text{hat} \\
&\text{show that} \\
&\text{show that} \\
&\text{flow } \uparrow \text{value.}\n\end{aligned}
$$

In this example, $f(w\cap z) = \frac{1}{2}y^{2} + \frac{1}{2}y^{3} - f(w) \wedge f(z)$

^① We want to show that $f(w\vee z)=f(w)\vee f(z)$ (\subseteq) : Let y \in f(WUZ). A $f(wvz)$ $e^+ y \in f(wvz)$
A
wuz
+ f FINDS OF CONSILLATION SWV く
そ G

Then there exists $x \in W \cup Z$ en There $EXISI$
Where $f(x) = y$.

Since XEWUZ we know X-W or XEZ.

So either YEf(w) or yef(z) from the two cases above. Thus, y So either yef(w) or yef(2)
from the two cases above.
Thus, yef(w) Uf(2).
(2): Let bef(w) Uf(2).
Then, bef(w) or bef(2).
case): Suppose bef(w). - f(w) Vf(z) . (2) : Let be $f(w) \cup f(z)$. Then, Then, $b \in f(w)$ or $b \in f(z)$.

Case): Suppose $b \in f(w)$.
 D there exists $\alpha \in W$ casel : Suppose bef(w). $\frac{1}{\sqrt{2}}$ $\frac{1}{\sqrt{2}}$ where $f(a) = b$
 $\frac{A}{2}$ B y E f (w) U f (Z).
Let b E f (w) U f (Z).
, b E f (w) or b E f (Z).
Suppose b E f (w).
there exists a E W
e f (a) = b B
A For the two cases above

S. Y E f (w) U f (Z).

: Let b E f (w) U f (Z).

n, b E f (w) or b E f (Z

1. Suppose b E f (w).

... there exists a E W

... there exists a E W

... A

... F (a) = b. B

... F (w)

... F (w)

... Theo there exi
Where $f(a) =$
(a) + $\frac{R}{1-\frac{1$ $\frac{1}{2}$ $\frac{1}{2}$

 But a \in W \subseteq W \cup Z. \int o, $a\in W\cup Z$ and $f(a)=b$. Thus , $b \in f(wvz)$. Cuse 2! Suppose WEF(Z). - Then there exists $a \in Z$

where $f(a)=b$.
 B
 $\left(\frac{f(wv)}{a}\right)$
 $f(wv)$
 $\left(\frac{f(wv)}{a}\right)$ $wheu f(a) = b$. β Then $\frac{1}{\mu}$
 $\frac{1}{\sqrt{2}}$ \overline{b} wUz and $t(e)$
 $b \in f(wUz)$.
 \vdots Suppose bet
 $f(a) = b$.
 a
 $b = x$ ov x , $\frac{f(z)}{f(z)}$ $BU+ACZ \le WUZ$, ζ \circ , $a \in W \cup Z$ and $f(a) = b$. Thus, $b\in f(W\cup Z)$.

There fore, in either case ^I or case 2 we get $b \in f(wvz)$. Thus, $f(w)\vee f(z) \subseteq f(w\vee z)$.

$$
By, (\le) and (\ge) we getf(wvz)=f(w)\cup f(\overline{z}).
$$

^④ Suppose WEZ. A Let ye f(w) , Then , there exists XEw with Since and f(x) WCZ xEW ⁼ y . ⑧

We know that XEZ. Since $x \in Z$ and $f(x) = y$ we know that yef(z). We know i.i. $y \in \{C\}$.
We have shown that $f(w) \subseteq f(z)$. We know that $x \in \mathbb{Z}$.
Since $x \in \mathbb{Z}$ and $f(x) = y$
we know that $y \in f(\mathbb{Z})$.
We have shown that $f(w) \subseteq f(\mathbb{Z})$ $E(f(z))$
 $+ f(w) 5f(z)$

Key: Recall: $x \in f^{-1}(w)$ $f: A \rightarrow B$ means $f(x) \in W$ $M \subseteq B$ $f^{-1}(W) = \begin{cases} x \in A & f(x) \in W \end{cases}$ $\overline{\mathcal{A}}$ \mathcal{M} $(0)^{1-}7$ $f(x)$ \times

Then : $O(f^{-1}(W\cap Z)) = f^{-1}(W)\cap f^{-1}(Z)$ $(3 + 1) (w \vee z) = f^{-1}(w) \vee f^{-1}(z)$ (3) A $f^{-1}(w) = f^{-1}(B-w)$ $9F + W \subseteq Z$, then $f'(W) \subseteq f'(z)$

proof:) Let's show that $f^{-1}(W\cap \Sigma)=f^{-1}(W)\cap f^{-1}(\Sigma).$ $\subset \mid$: Let a E f⁻¹ (WNZ). K W $f^{-1}(W()2)$ Then, f(a) EWNZ. $S_0, f(a) \in W$ and $f(a) \in Z$.

Thus, $a\in f^{-1}(w)$ and $a\in f^{-1}(\mathcal{Z})$. Therefore, $a \in f^{-1}(w) \cap f^{-1}$ (7) Thus, $\alpha \in f^{-1}(w)$ and $\alpha \in f^{-1}$

Therefore, $\alpha \in f^{-1}(w) \cap f^{-1}(\overline{z})$
 $\boxed{\supseteq}$:

Let $x \in f^{-1}(w) \cap f^{-1}(\overline{z})$ Let $xe + (W)$ 111 (27)
Then, $xe + (W)$ and $xe + (Z)$. S_{0} , $f(x)\in W$ and $f(x)\in Z$ Thus, $f(x) \in W \cap Z$. \mathcal{S} o, $x\in f^{-1}(W\cap Z).$ $By (S)$ and (2) we get $f^{-1}(w\wedge z)=f^{-1}(w)\wedge f^{-1}(z).$

(2) Let's show that $f^{-1}(WUZ) = f^{-1}(W)Uf^{-1}(Z)$ $\vert \subseteq \vert$: Let a∈f^{-'}(WVZ). $f^{-1}(WU)$ $f(a)$

 $S_0, f(a) \in WUZ.$
Thus, $f(a) \in W$ or $f(a) \in Z$. Hence, $f(n)$ even neared.
a e f ' (w) or a e f ' (z). $Ergo,$ $a\in f^{-1}(W)\cup f^{-1}(\mathcal{Z}).$ E rgo, $a \in f^{-1}(W) \cup f^{-1}(Z)$.
Thus, $f^{-1}(W \cup Z) \subseteq f^{-1}(W) \cup f^{-1}(Z)$. Hend
 Eig
 $\frac{1}{2}$:
 e $[2]^{\cdot}$ Let $x \in f^{-1}(W) \cup f^{-1}(z)$. Then, $xef^{-1}(w)$ or $f^{-1}(z)$. So, r, xer (vo)
f(x) E W ur f(x) EZ. Hence, $f(x)\in WUZ$. $TNV5$ Xef"(WVZ1 . Hence, $f^{-1}(w) \cup f^{-1}(z) \subseteq f^{-1}(w \cup z)$

 $By \in)$ and (2) we have $f_{\text{tot}}(z) = f_{\text{tot}}(w) \cup f_{\text{tot}}(z)$

Ift version of (2): $By (S) and (2) we have
\n+hat f''(wvz) = f'(w)Vf''(z)$
 $+hat f'''(wvz) = f'(w)Vf''(z)$
 $\overline{C} = f^{-1}(wVz)$
 $\overline{C} = f^{-1}(wVz)$
 $\overline{C} = f^{-1}(wVz)$
 $\overline{C} = f^{-1}(wVz)$
 $\overline{C} = f^{-1}(wVz) + f^{-1}(z)$
 $\overline{C} = f^{-1}(wVz) + f^{-1}(z)$
 $\overline{C} = f^{-1}(wVz) + f^{-1}(z)$ $a\in f^{-1}(W\cup Z)$ $iff f(a) \in WVZ$ iff $f(a)\in W$ or $f(a)\in Z$ iff $a \in f^{-1}(w)$ or $a \in f^{-1}(z)$ iff $a \in f$ (w) v.

iff $a \in f$ (w) Uf " (7) $iff \alpha \in f^{-1}(W) \cup f^{-1}(Z)$
Thus, $f^{-1}(W \cup Z) = f^{-1}(W) \cup f^{-1}(Z)$ $2y(\subseteq)$ and (2) we have
 $+hat f^{-1}(wvz) = f^{-1}(w) vf^{-1}(z)$
 $\frac{1}{\pi}$
 $\frac{1}{\pi}$ version of (2):
 $\alpha \in f^{-1}(wvz)$

iff $f(\alpha) \in Wvz$

iff $f(\alpha) \in Wvz$

iff $f(\alpha) \in Wvz$

iff $\alpha \in f^{-1}(w)$ or $\alpha \in f^{-1}(z)$

iff $\alpha \in f^{-1}(w) vf^{-1}(z)$

Th

$$
(3) Let's show that\nA-f'(w) = f'(B-w)\nWe have that $a \in A - f^{-1}(W)$
\nif f a \in A and $a \notin f'(W)$
\n $f^{-1}(W)$
\n $f^{-1}(W)$
\n $f^{-1}(W)$
\n $f^{-1}(W)$
\nif f(a) \in B - W.
\nif f(a) \in B - W.
\n $f^{+1}(W) = f^{-1}(B-W)$.
\nThus, A-f'(w) = f^{-1}(B-W).
$$

 (4) Suppose that $W \subseteq Z$. Let's prove that $f^{-1}(w) \subseteq f^{-1}(z)$ Let a E f⁻¹(W).

Then, $f(a) \in W$. Since f(a) EW and W = Z, we Know that $f(a) \in Z$. Thus, $a \in f^{-1}(\mathcal{Z}).$ Hence, $f^{-1}(w) \subseteq f^{-1}(Z)$.