Topic 4 - Functions

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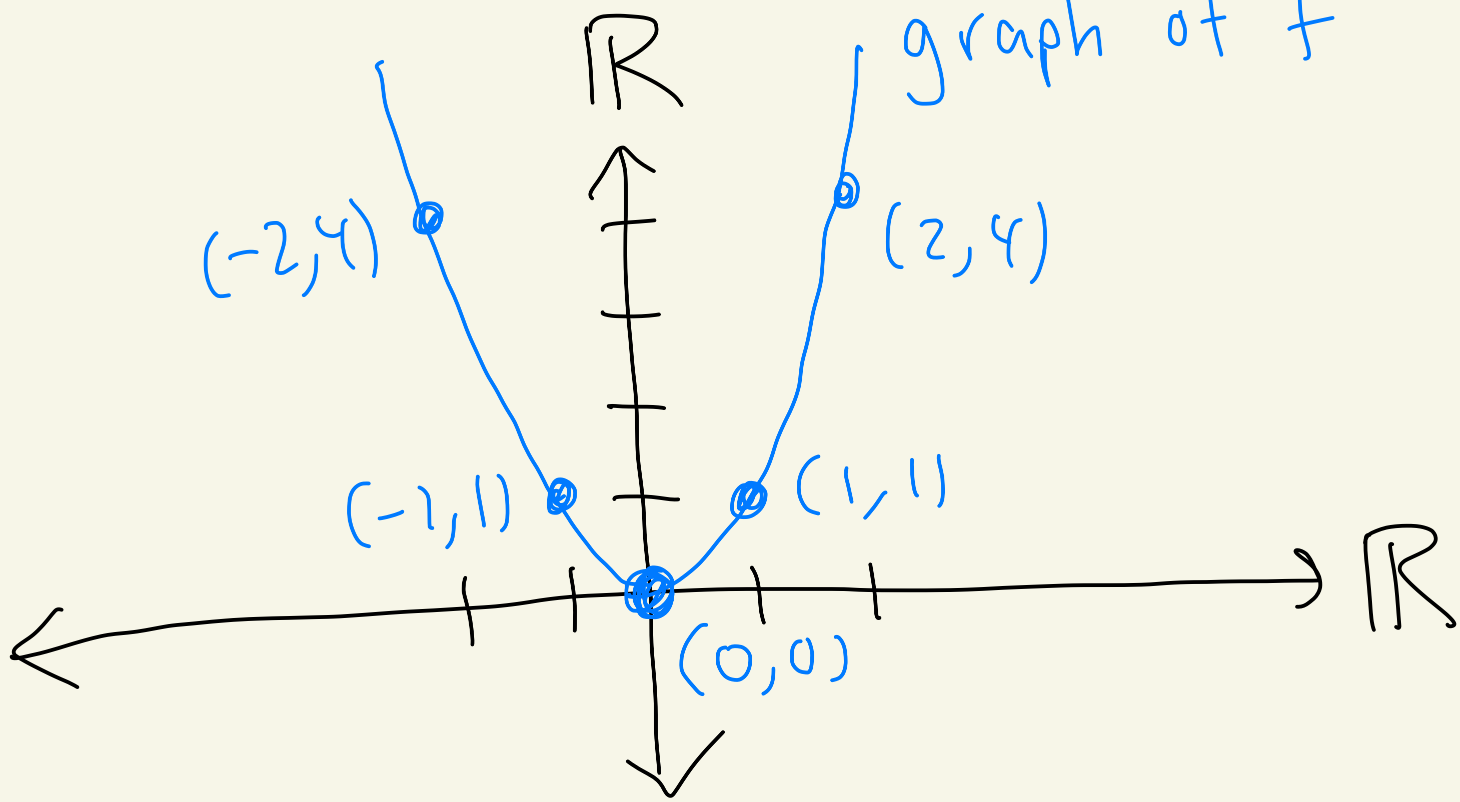
We are going to formally define functions as sets but then after that we won't really use that method anymore we will just use formulas like usual.

<u>Ex:</u> Consider the function $f(x) = x^2$ where $x \in \mathbb{R}$.



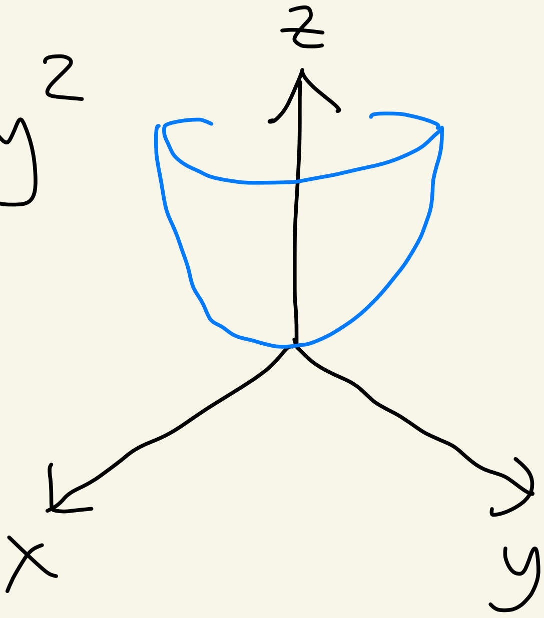
The graph is

$$\{(x, x^2) \mid x \in \mathbb{R}\}$$

This graph lives inside of

$$\underbrace{\mathbb{R}}_{\text{domain}} \times \underbrace{\mathbb{R}}_{\text{co-domain, where the range lives}}$$

Ex: $f(x,y) = x^2 + y^2$



graph lives in

$$\mathbb{R}^3 = \underbrace{\mathbb{R} \times \mathbb{R}}_{\text{domain}} \times \underbrace{\mathbb{R}}_{\text{co-domain}}$$

Def: Let A and B be sets. Let f be a subset of $A \times B$. We say that **f is a function from A to B** if

1. for every $a \in A$ there exists $b \in B$ where $(a,b) \in f$

this is saying that we can plug a into f to get b, ie $f(a)=b$

and

2. if $(a,b_1) \in f$ and $(a,b_2) \in f$, then $b_1 = b_2$

vertical line test

If this is the case then we write $f: A \to B$ to mean that f is a function from A to B

The set A is called the
<u>domain</u> of f.

The set B is called the
<u>co-domain</u> of f.

If $(a,b) \in f$ then
we write $f(a) = b$

The <u>range</u> of f is

$$\text{range}(f) = \left\{ b \in B \;\middle|\; \begin{array}{l} \text{there exists } a \in A \\ \text{with } f(a) = b \end{array} \right\}$$

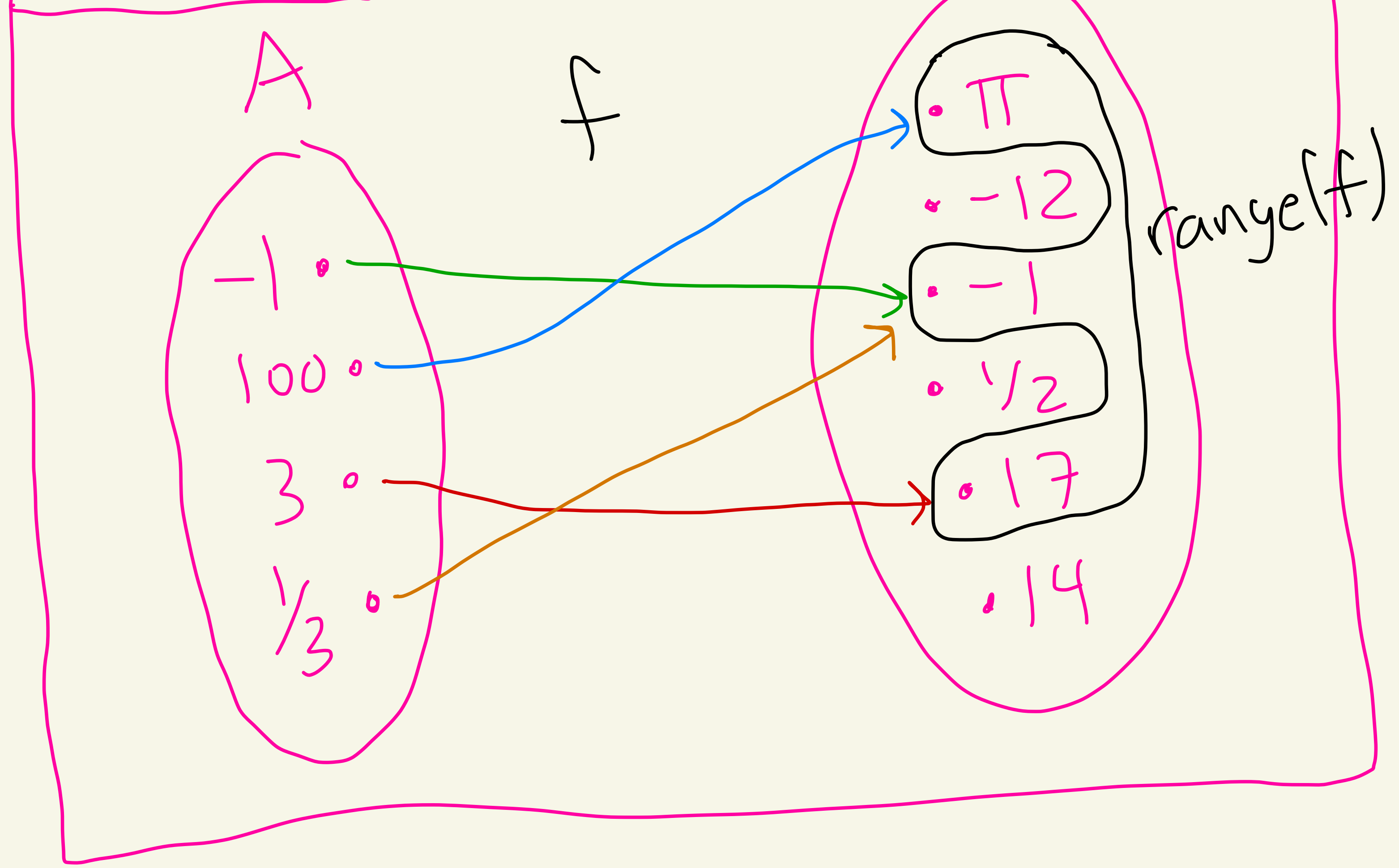
Ex: $A = \{-1, 100, 3, \frac{1}{3}\}$

$B = \{\pi, -12, -1, \frac{1}{2}, 17, 14\}$

$f = \{(-1,-1), (100,\pi), (3,17), (\frac{1}{3},-1)\}$

$f(-1) = -1$ $\quad$ $f(100) = \pi$ $\quad$ $f(3) = 17$ $\quad$ $f(\frac{1}{3}) = -1$

picture



Is f a function from A to B?

1. f is defined on all of A ✓
2. no element of A gets mapped to more than one element of B ✓

Yes, f is a function from A to B.

$$\text{domain}(f) = A$$

$$\text{co-domain}(f) = B$$

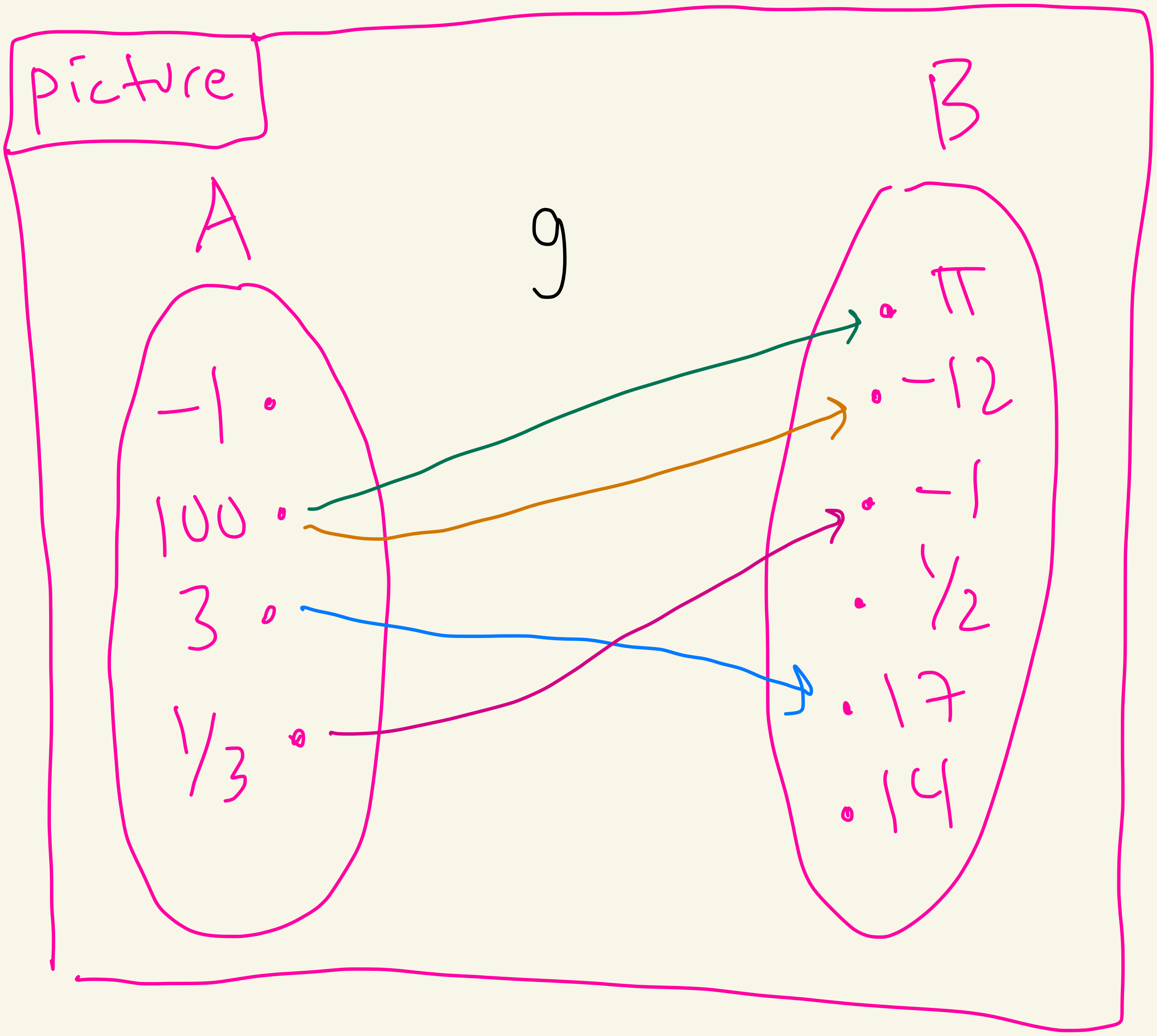
$$\text{range}(f) = \underbrace{\{\pi, -1, 17\}}_{\text{subset of co-domain } B}$$

<u>Ex:</u> $A = \{-1, 100, 3, \frac{1}{3}\}$

$B = \{\pi, -12, -1, \frac{1}{2}, 17, 14\}$

$g = \{(100, \pi), (3, 17), (\frac{1}{3}, -1), (100, -12)\}$

$g(100) = \pi$ $\quad$ $g(3) = 17$ $\quad$ $g(\frac{1}{3}) = -1$ $\quad$ $g(100) = -12$



① $g(-1)$ is not defined

2. $g(100)$ has two values: π & -12 X

g is not a function from A to B.

Let's now use formulas to define functions instead of defining them as subsets of $A \times B$.

Ex: Let A be any non-empty set. The identity function on A is the function

$$i_A : A \to A$$

defined as

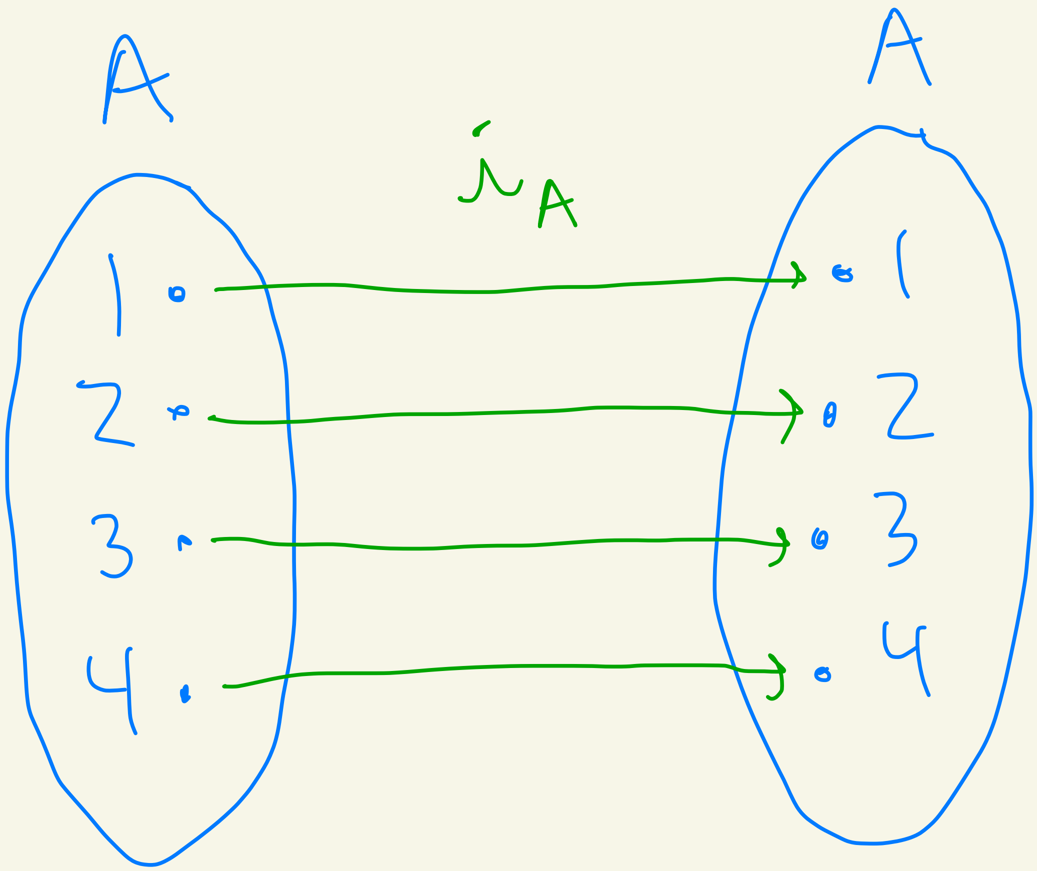
$$i_A(x) = x \quad \text{for all } x \in A.$$

Sometimes we will just write i instead of i_A.

Formally you can think of

$$i_A = \{ \underbrace{(x,x)}_{i_A(x)=x} \mid x \in A \} \subseteq A \times A$$

Ex: $A = \{1, 2, 3, 4\}$



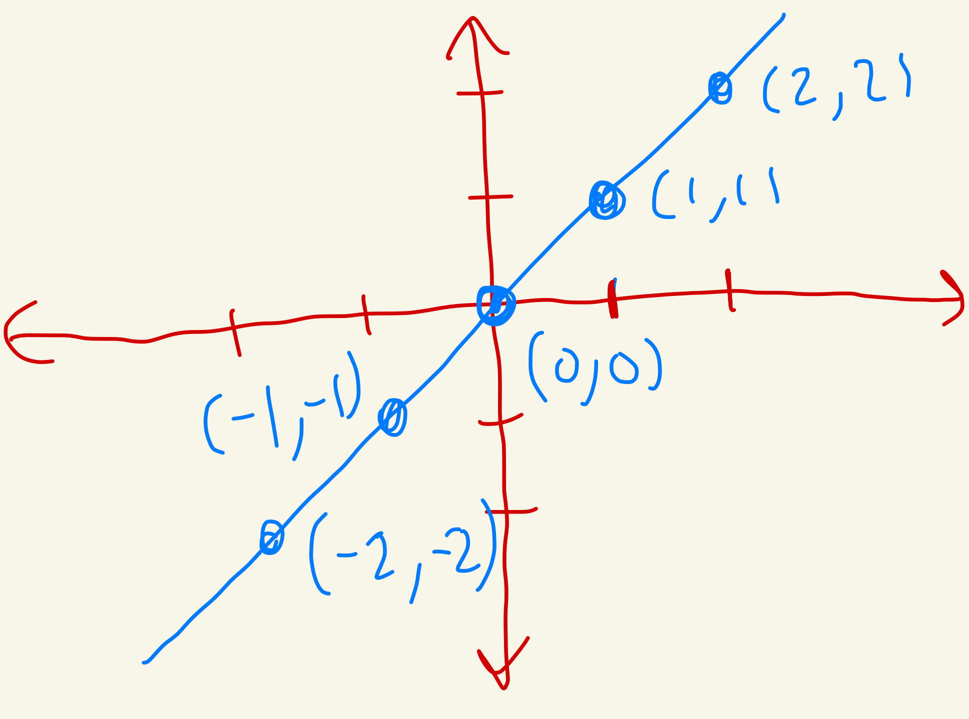
$i_A(1) = 1$

$i_A(2) = 2$

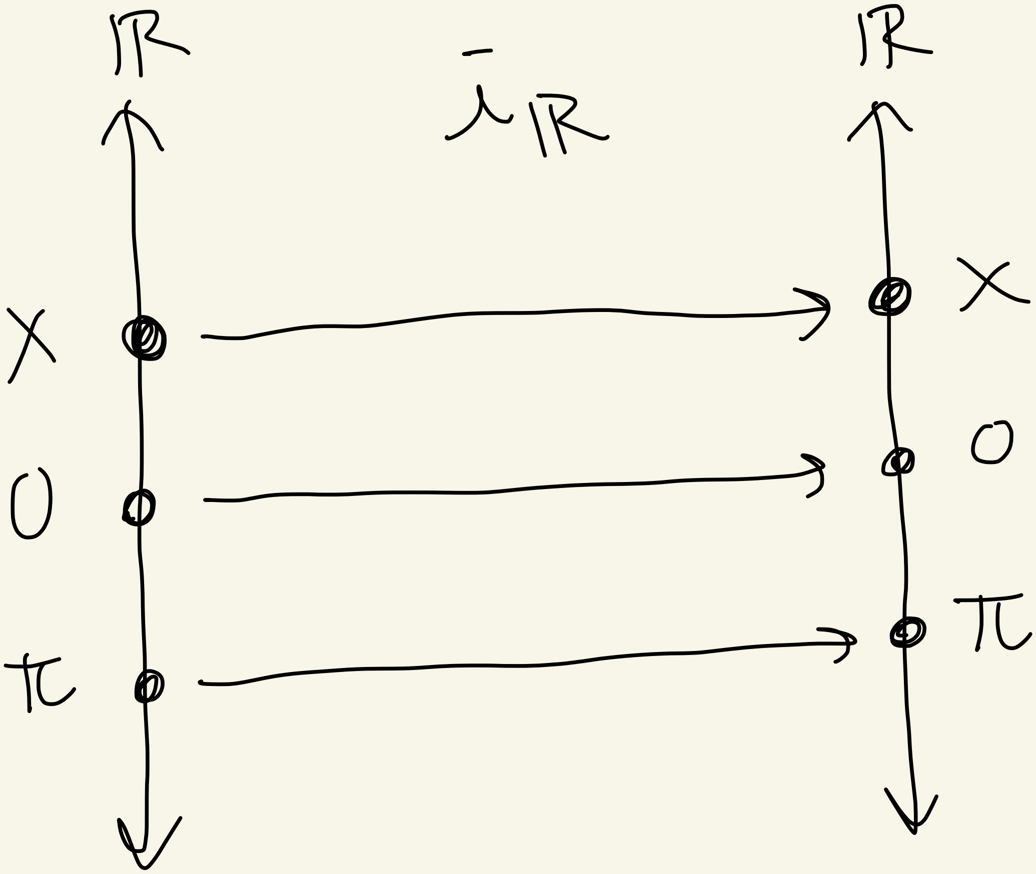
$i_A(3) = 3$

$i_A(4) = 4$

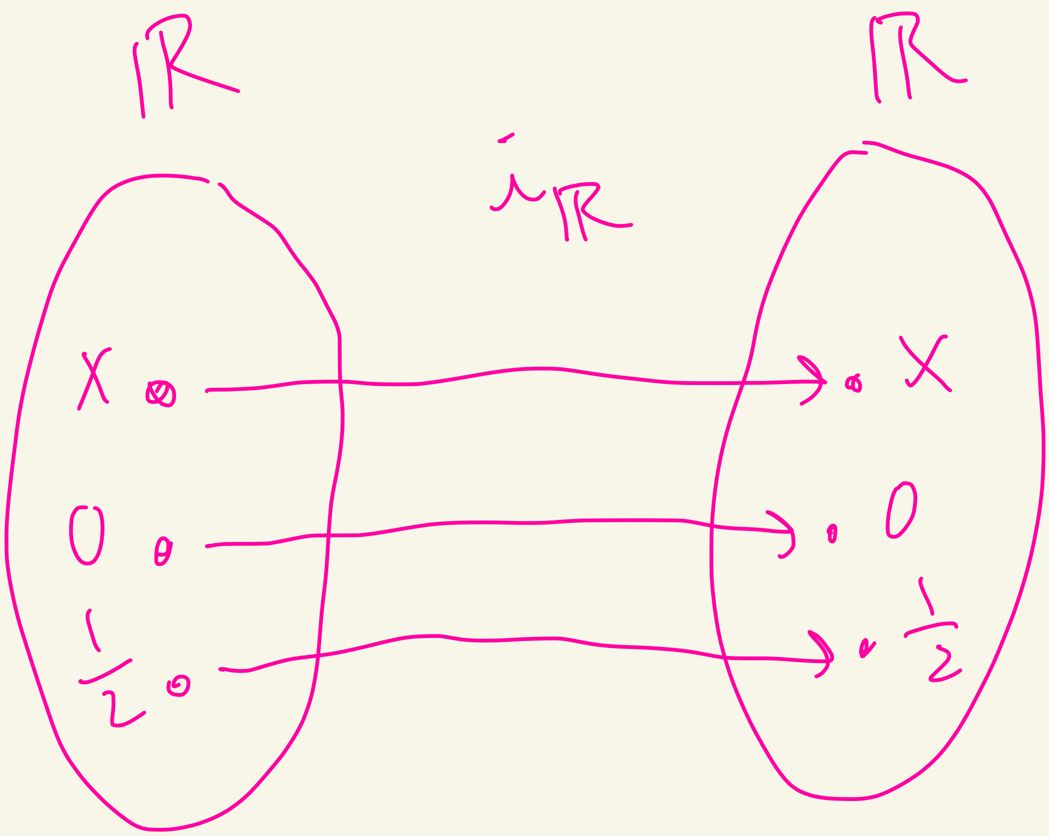
Ex: $A = \mathbb{R}$, $i_{\mathbb{R}} : \mathbb{R} \to \mathbb{R}$, $i_{\mathbb{R}}(x) = x$



graph way to draw $i_{\mathbb{R}}$



another way to draw



another way to draw

Ex: Let $n \in \mathbb{Z}$, $n \geq 2$.

Define the <u>reduction modulo n map</u> to be

map is another name for function some use mapping

$$\pi_n : \mathbb{Z} \longrightarrow \mathbb{Z}_n$$

where $\pi_n(x) = \overline{x}$

Ex: $n = 3$

$$\mathbb{Z}_3 = \{\overline{0}, \overline{1}, \overline{2}\}$$

$$\pi_3 : \mathbb{Z} \to \mathbb{Z}_3, \quad \pi_3(x) = \overline{x}$$

some computations are:

$\pi_3(0) = \overline{0}$

$\pi_3(1) = \overline{1}$

$\pi_3(-1) = \overline{-1} = \overline{2}$

$\pi_3(-2) = \overline{-2} = \overline{1}$

$\pi_3(2) = \overline{2}$

$\pi_3(3) = \overline{3} = \overline{0}$

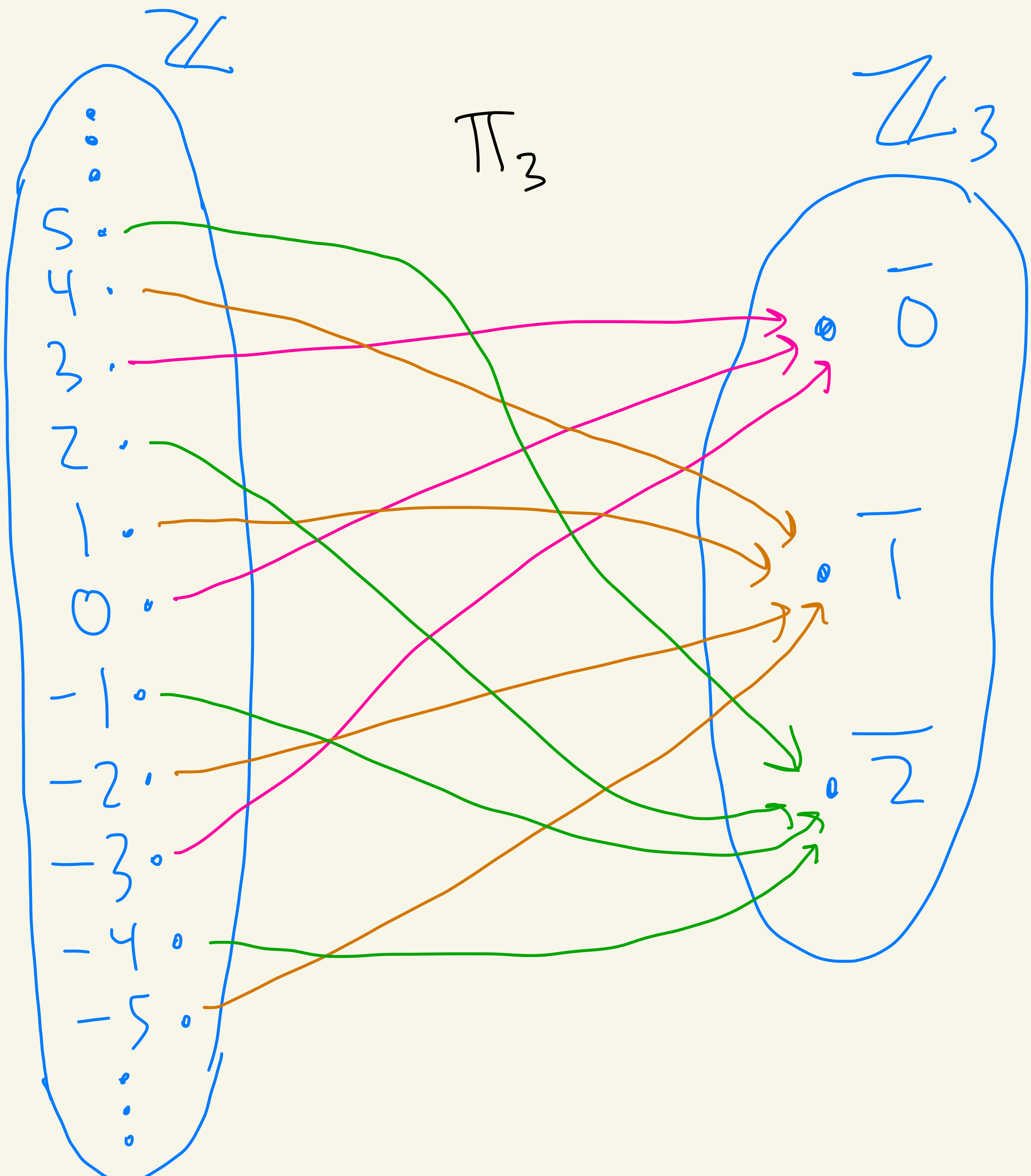
$\pi_3(4) = \overline{4} = \overline{1}$

$\pi_3(5) = \overline{5} = \overline{2}$

$\pi_3(-3) = \overline{-3} = \overline{0}$

$\pi_3(-4) = \overline{-4} = \overline{2}$

$\pi_3(-5) = \overline{-5} = \overline{1}$



$\mathrm{domain}(\pi_3) = \mathbb{Z}$

$\text{co-domain}(\pi_3) = \mathbb{Z}_3$

$\mathrm{range}(\pi_3) = \{\bar{0}, \bar{1}, \bar{2}\} = \mathbb{Z}_3$

$\boxed{\text{Well-defined functions}}$

Ex: Suppose you and your friend Francis want to define a function on $\mathbb{Q}$. You say "How about this function? $f:\mathbb{Q}\to\mathbb{Q}$ where $f\left(\frac{a}{b}\right)=\frac{b}{a}$"

Francis says "I don't know about that function. What about $f\left(\frac{0}{1}\right)=\frac{1}{0}$? That doesn't seem to make sense."

You say "you're right. good call."

Then you say, "Ok I've got

another idea. How about $g: \mathbb{Q} \to \mathbb{Q}$ where $g\left(\frac{a}{b}\right) = a$?

That totally works. For example, $g\left(\frac{3}{5}\right) = 3$ and $g\left(\frac{0}{2}\right) = 0$."

Then Francis says, "Hey wait a minute, $g\left(\frac{3}{5}\right) = 3$ but $g\left(\frac{6}{10}\right) = 6$ and $\frac{3}{5} = \frac{6}{10}$. Shouldn't g agree on those numbers?"

You say "oh yeah you're right."

The functions f and g above are not well-defined.

How to check that $f: A \to B$ is well-defined

Check two things:

1. If $a \in A$, then $f(a) \in B$
2. If some or all of the elements from A can be expressed in more than one way then we must check that if a_1, a_2 are two expressions of the same element in A (ie $a_1 = a_2$) then $f(a_1) = f(a_2)$

Ex: Let $f: \mathbb{Q} \to \mathbb{Q}$ where $f\left(\frac{a}{b}\right) = \left(\frac{a}{b}\right)^2$.

Is f well-defined? Yes!

proof that f is well-defined:

① Let $\frac{a}{b} \in \mathbb{Q}$.

So, $a, b \in \mathbb{Z}$ and $b \neq 0$.

Then, $f\left(\frac{a}{b}\right) = \left(\frac{a}{b}\right)^2 = \frac{a^2}{b^2}$

We have that $a^2, b^2 \in \mathbb{Z}$ and $b^2 \neq 0$ (since $b \neq 0$).

So, $\frac{a^2}{b^2} \in \mathbb{Q}$.

② Suppose $\frac{a}{b}, \frac{c}{d} \in \mathbb{Q}$ and $\frac{a}{b} = \frac{c}{d}$.

Is $f\left(\frac{a}{b}\right) = f\left(\frac{c}{d}\right)$?

Method 1:

Since $\frac{a}{b} = \frac{c}{d}$, then by squaring both sides we get $\left(\frac{a}{b}\right)^2 = \left(\frac{c}{d}\right)^2$.

So, $f\left(\frac{a}{b}\right) = f\left(\frac{c}{d}\right)$

You might ask, why is this true?

Method 2:

Recall how we define two fractions to be equal:

$$\frac{w}{x} = \frac{y}{z} \quad \text{means} \quad wz = xy$$

Suppose $\frac{a}{b} = \frac{c}{d}$.

Then $ad = bc$.

So, $(ad)^2 = (bc)^2$ using integer mult. is well-defined

Then, $a^2 d^2 = b^2 c^2$

So, $\frac{a^2}{b^2} = \frac{c^2}{d^2}$

Thus, $f\left(\frac{a}{b}\right) = f\left(\frac{c}{d}\right)$

From 1 and 2 above

f is well-defined. ∎

Ex: Let $n \in \mathbb{Z}$, $n \geq 2$.

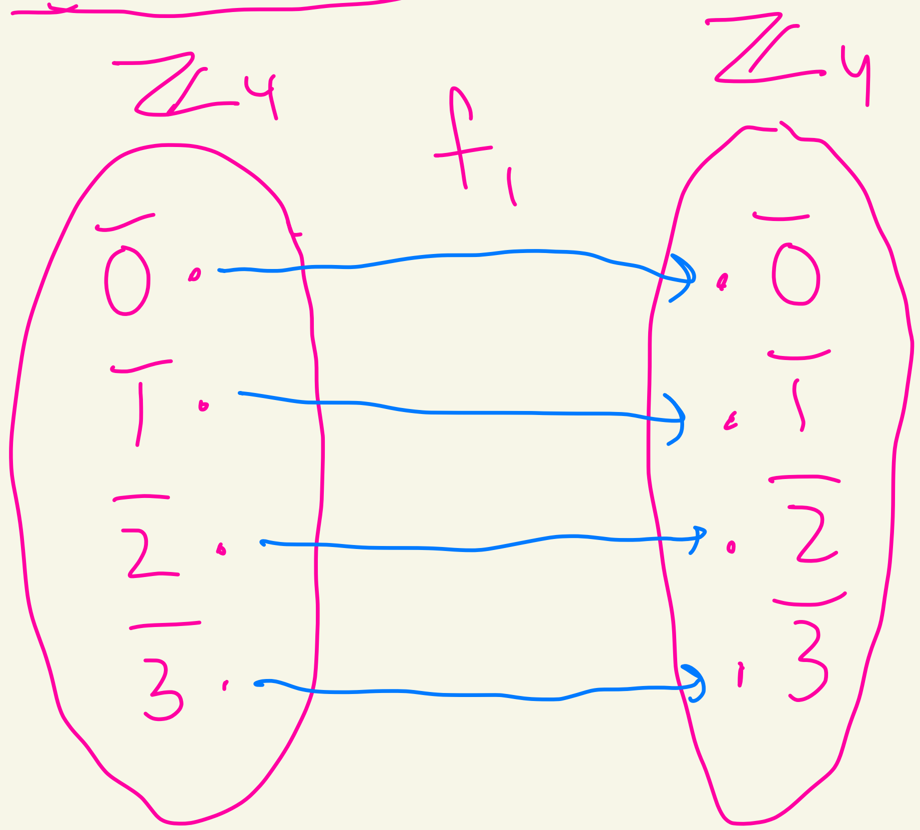
Pick $a \in \mathbb{Z}$.

Define $f_a : \mathbb{Z}_n \to \mathbb{Z}_n$

by $f_a(\bar{x}) = \bar{a} \cdot \bar{x}$

Let's do some examples

when $n = 4$, $\mathbb{Z}_4 = \{\bar{0}, \bar{1}, \bar{2}, \bar{3}\}$

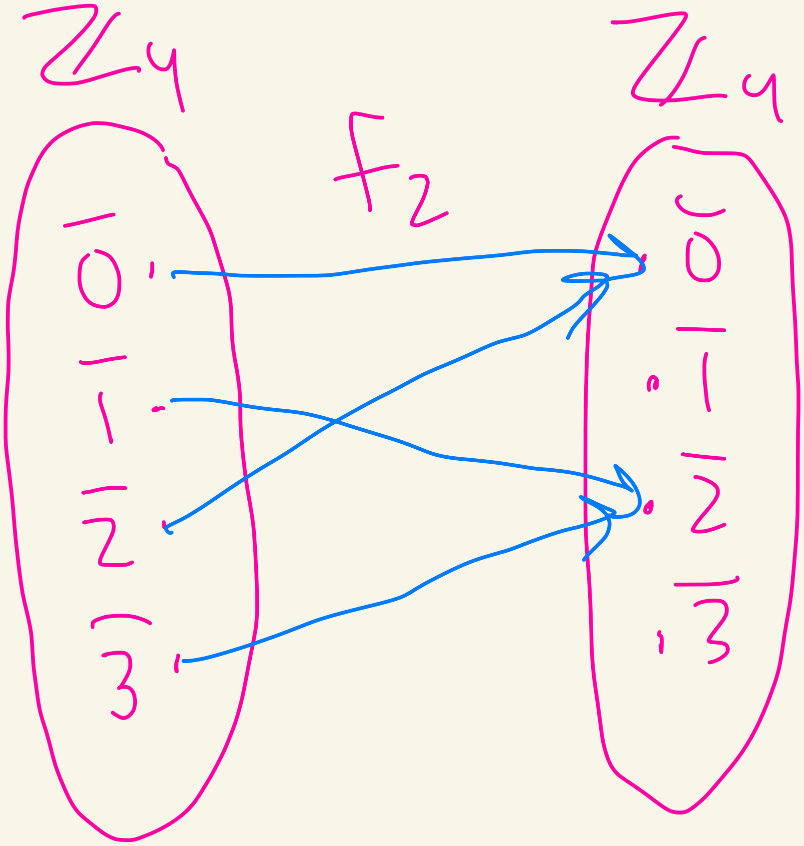


$f_1(\bar{0}) = \bar{1} \cdot \bar{0} = \bar{0}$

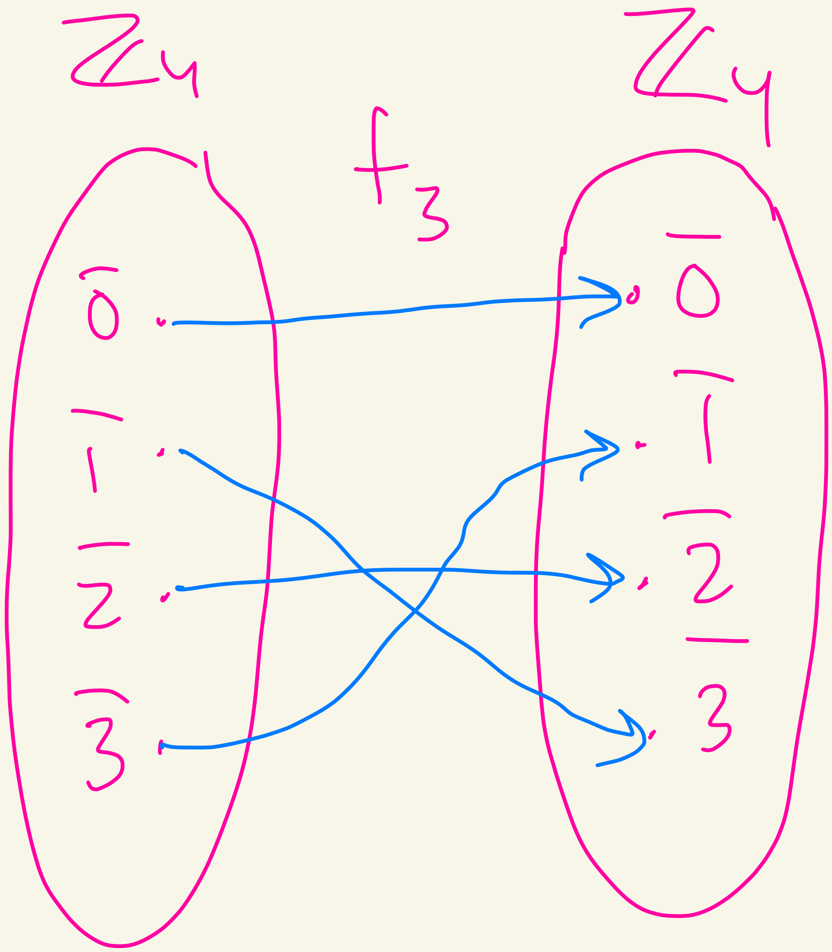
$f_1(\bar{1}) = \bar{1} \cdot \bar{1} = \bar{1}$

$f_1(\bar{2}) = \bar{1} \cdot \bar{2} = \bar{2}$

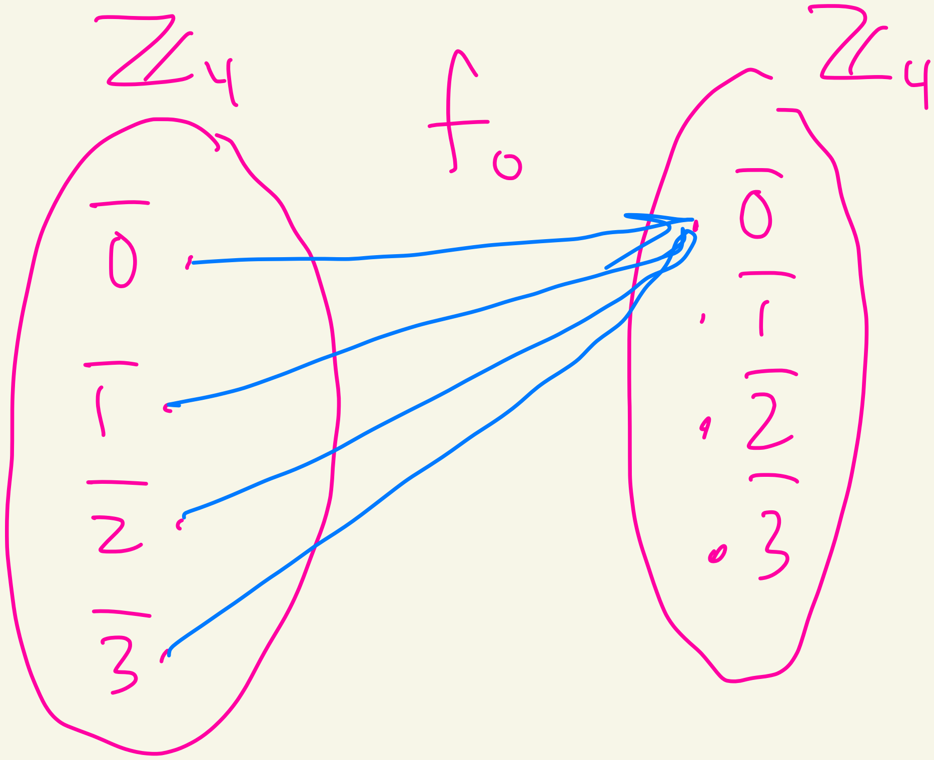
$f_1(\bar{3}) = \bar{1} \cdot \bar{3} = \bar{3}$



$$f_2(\bar{0}) = \bar{2}\cdot\bar{0} = \bar{0}$$
$$f_2(\bar{1}) = \bar{2}\cdot\bar{1} = \bar{2}$$
$$f_2(\bar{2}) = \bar{2}\cdot\bar{2} = \bar{4} = \bar{0}$$
$$f_2(\bar{3}) = \bar{2}\cdot\bar{3} = \bar{6} = \bar{2}$$



$$f_3(\bar{0}) = \bar{3}\cdot\bar{0} = \bar{0}$$
$$f_3(\bar{1}) = \bar{3}\cdot\bar{1} = \bar{3}$$
$$f_3(\bar{2}) = \bar{3}\cdot\bar{2} = \bar{6} = \bar{2}$$
$$f_3(\bar{3}) = \bar{3}\cdot\bar{3} = \bar{9} = \bar{1}$$



$$f_0(\bar{x}) = \bar{0}\cdot\bar{x}$$
$$= \bar{0}$$

Theorem: Let $n \in \mathbb{Z}$, $n \geq 2$. Let $a \in \mathbb{Z}$. Let $f_a : \mathbb{Z}_n \to \mathbb{Z}_n$ be given by $f_a(\bar{x}) = \bar{a}\cdot\bar{x}$. Then f_a is well-defined.

proof:

① Let $\bar{x} \in \mathbb{Z}_n$ where $x \in \mathbb{Z}$.

Since $x, a \in \mathbb{Z}$ we know $ax \in \mathbb{Z}$.

Thus,

$$f_a(\overline{x}) = \overline{a} \cdot \overline{x} = \overline{ax} \in \mathbb{Z}_n.$$

(2) Let $\overline{x}, \overline{y} \in \mathbb{Z}_n$ where $\overline{x} = \overline{y}$.

Then,

$$f_a(\overline{x}) = \overline{a} \cdot \overline{x} \overset{\text{since } \overline{x}=\overline{y}}{=} \overline{a} \cdot \overline{y} = f_a(\overline{y}).$$

when we talked about well-defined operations we proved that if $\overline{b} = \overline{c}$ and $\overline{d} = \overline{e}$, then $\overline{b} \cdot \overline{d} = \overline{c} \cdot \overline{e}$

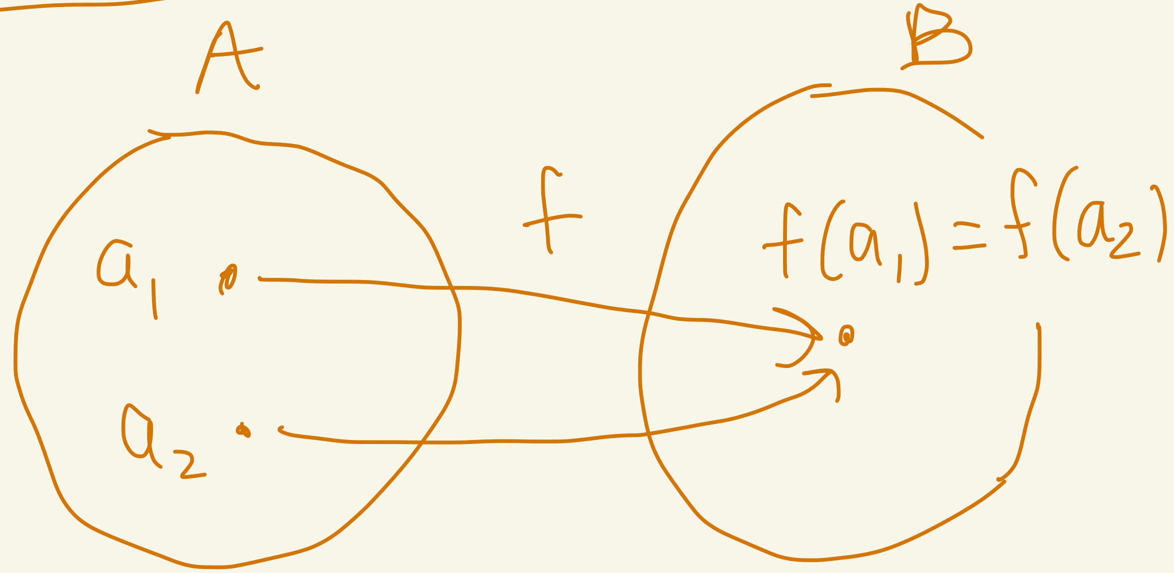
$\blacksquare$

Def: Let A and B be sets.

Let $f : A \to B$ be a function.
We say that f is *injective* or *one-to-one* if the following is true:

For all $a_1, a_2 \in A$,
if $a_1 \neq a_2$, then $f(a_1) \neq f(a_2)$

Ie you cannot have this



Another way to define:

For all $a_1, a_2 \in A$,

if $f(a_1) = f(a_2)$, then $a_1 = a_2$

How to prove $f: A \to B$ is one-to-one

Let $a_1, a_2 \in A$.

Suppose $f(a_1) = f(a_2)$

$\vdots$ (proof stuff)

conclude $a_1 = a_2$

Ex: Let $f: \mathbb{R} \to \mathbb{R}$ be defined by $f(x) = -4x+5$

Let's prove f is one-to-one.

pf: Suppose $x_1, x_2 \in \mathbb{R}$ and $f(x_1) = f(x_2)$.

Then, $-4x_1 + 5 = -4x_2 + 5$.

-5

Thus, $-4x_1 = -4x_2$.

$\times\left(-\frac{1}{4}\right)$

So, $x_1 = x_2$.

Thus, f is one-to-one

$\blacksquare$

How to show $f: A \to B$ is not one-to-one

Find specific $x_1, x_2 \in A$
where $x_1 \neq x_2$
but $f(x_1) = f(x_2)$

Ex: Let $n \in \mathbb{Z}$, $n \geq 2$.
Define $f: \mathbb{Z}_n \to \mathbb{Z}_n$
by $f(\overline{x}) = (\overline{x})^2$.

Claim: f is well-defined.

pf of claim:

① Given $\overline{x} \in \mathbb{Z}_n$ where $x \in \mathbb{Z}$ we have that

$$f(\overline{x}) = \overline{x}^2 = \overline{x} \cdot \overline{x} = \overline{x^2}.$$

Since $x \in \mathbb{Z}$ we know $x^2 \in \mathbb{Z}$. Thus, $f(\overline{x}) = \overline{x^2} \in \mathbb{Z}_n$.

② Suppose $\overline{x_1}, \overline{x_2} \in \mathbb{Z}_n$ and $\overline{x_1} = \overline{x_2}$. Then,

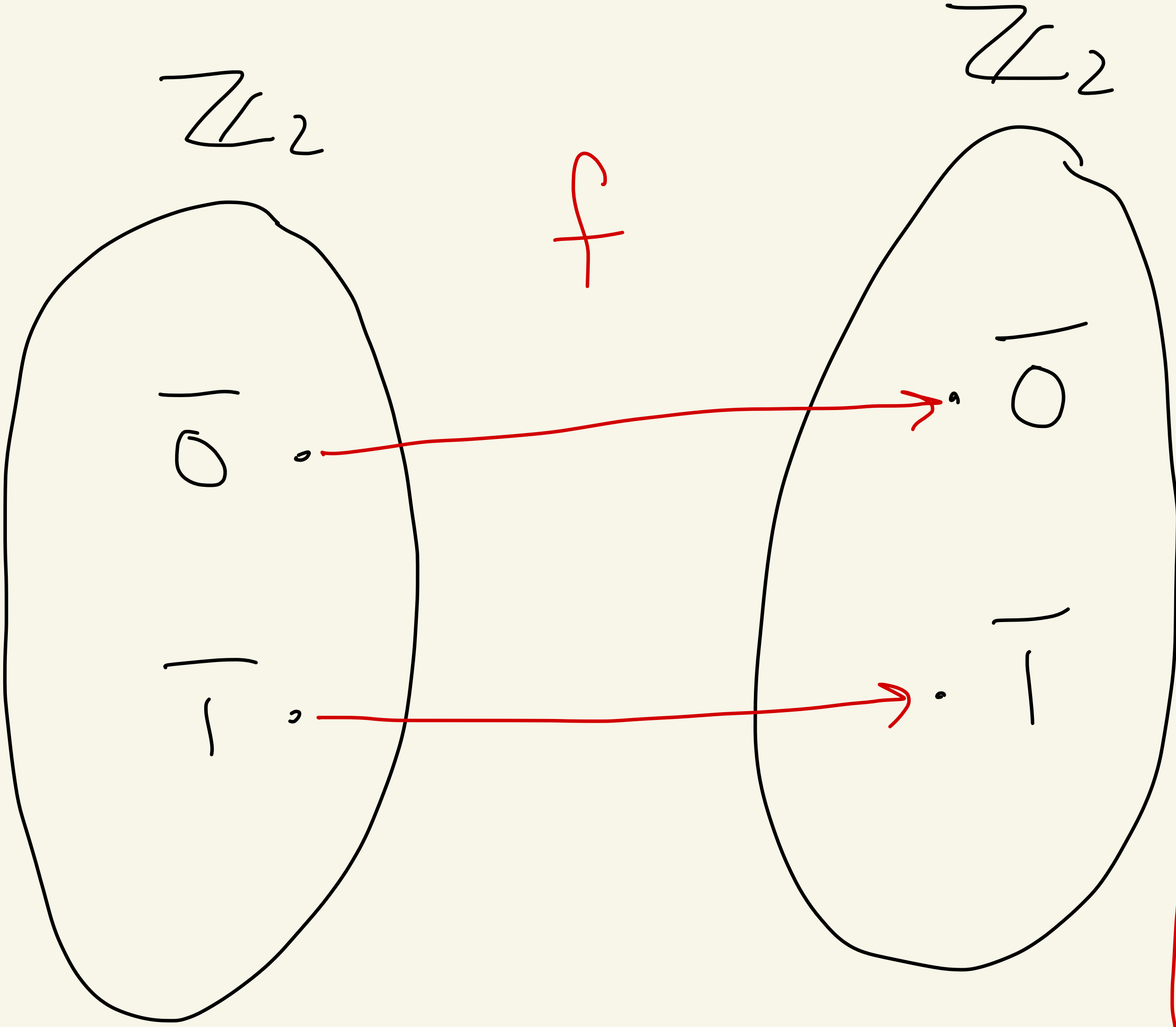
$$f(\overline{x_1}) = \overline{x_1}^2 = \overline{x_2}^2 = f(\overline{x_2})$$

mult. is well-defined in $\mathbb{Z}_n$, if $\overline{a} = \overline{c}$ and $\overline{b} = \overline{d}$, then $\overline{a}\,\overline{b} = \overline{c}\,\overline{d}$

Use with $\overline{a} = \overline{b} = \overline{x_1}$ and $\overline{c} = \overline{d} = \overline{x_2}$

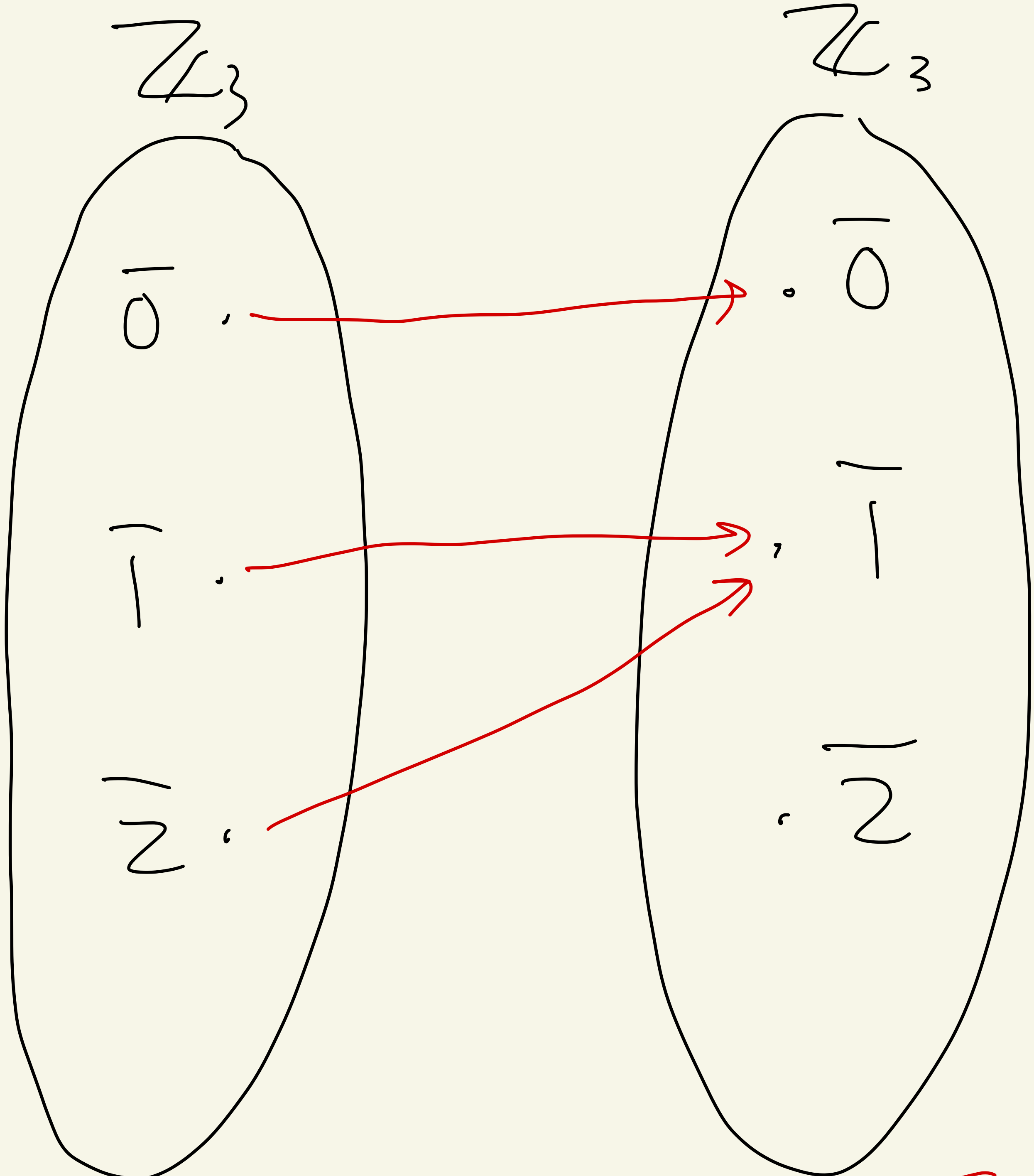
claim

$\boxed{\text{Ex:}}$ $n=2$, $f(\bar{x}) = \bar{x}^2$



$\boxed{f \text{ is } 1\text{-}1}$

Ex: $n=3$, $f(\bar{x})=\bar{x}^2$



$\boxed{f(\bar{2})=\bar{4}=\bar{1}}$

$\boxed{\bar{2}=\overline{-1} \;\&\; (\overline{-1})^2=\bar{1}}$

$\boxed{f \text{ is not } 1\text{-}1}$

Claim: Let $f: \mathbb{Z}_n \to \mathbb{Z}_n$ given by $f(\overline{x}) = \overline{x}^2$. If $n > 2$, then f is not one-to-one.

proof of claim:

Note first that since $n > 2$ we know that $\overline{1} \neq \overline{-1}$

$\boxed{\text{Why?}}$ Suppose $\overline{1} = \overline{-1}$.

Then, $1 \equiv -1 \pmod{n}$.

Thus, $n \mid (1-(-1))$

Ie, $n \mid 2$.

Thus, $n = \pm 1, \pm 2$.

Can't happen since $n > 2$

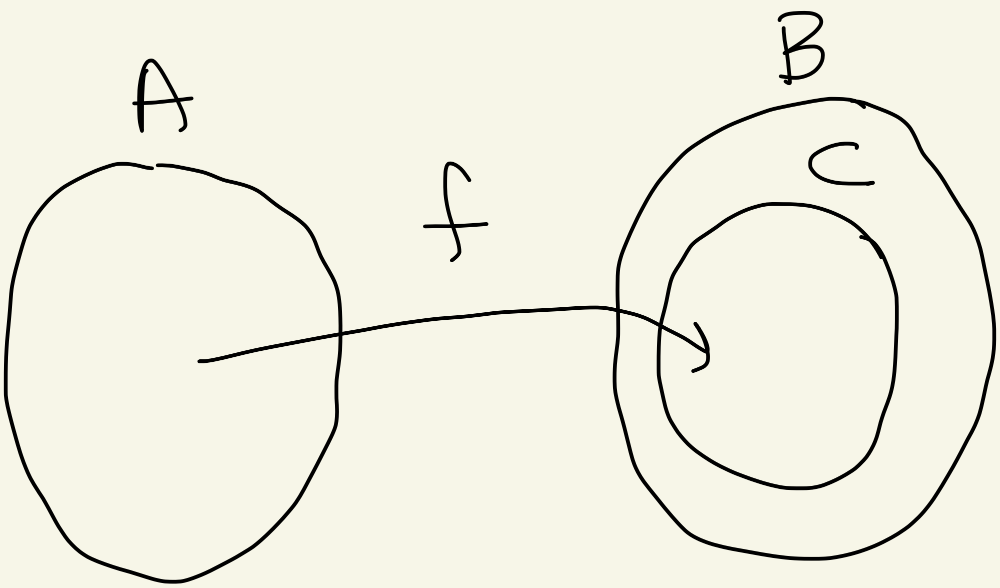
Thus, $\overline{1} \neq \overline{-1}$, however

$f(\overline{1}) = \overline{1}^2 = \overline{1}$ and

$f(\overline{-1}) = \overline{-1}^2 = \overline{1}$.

So, f is not 1-1 if $n > 2$. $\blacksquare$

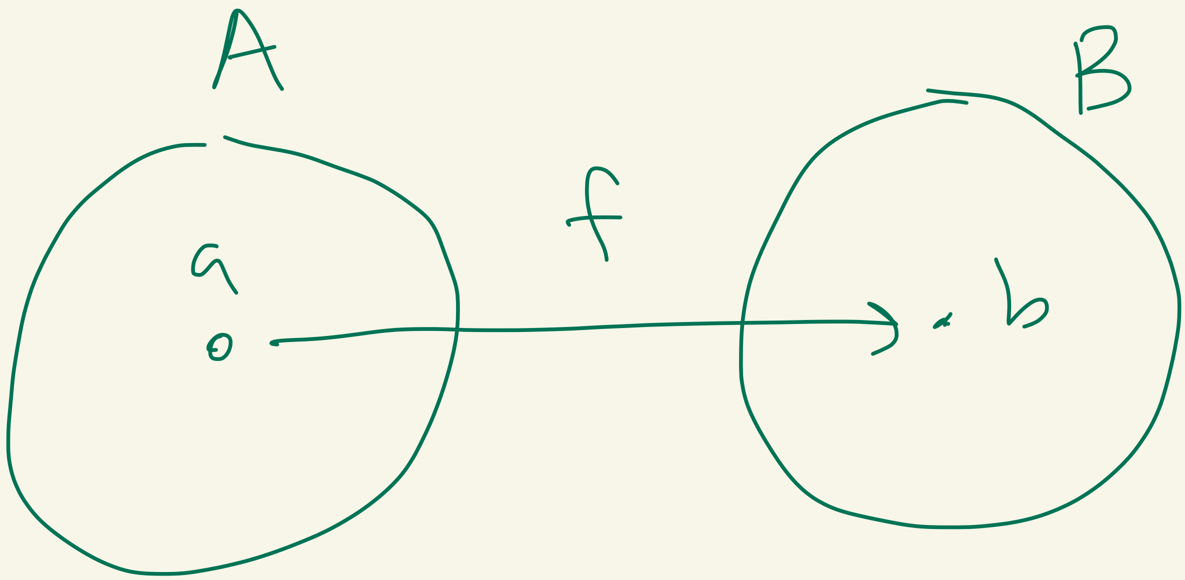
__Def:__ Let A and B be sets. Let $f: A \to B$. Let C be the range of f. We say that f is __surjective__ or __onto__ B if $C = B$.



f is onto if $C = B$

__Another way to say:__

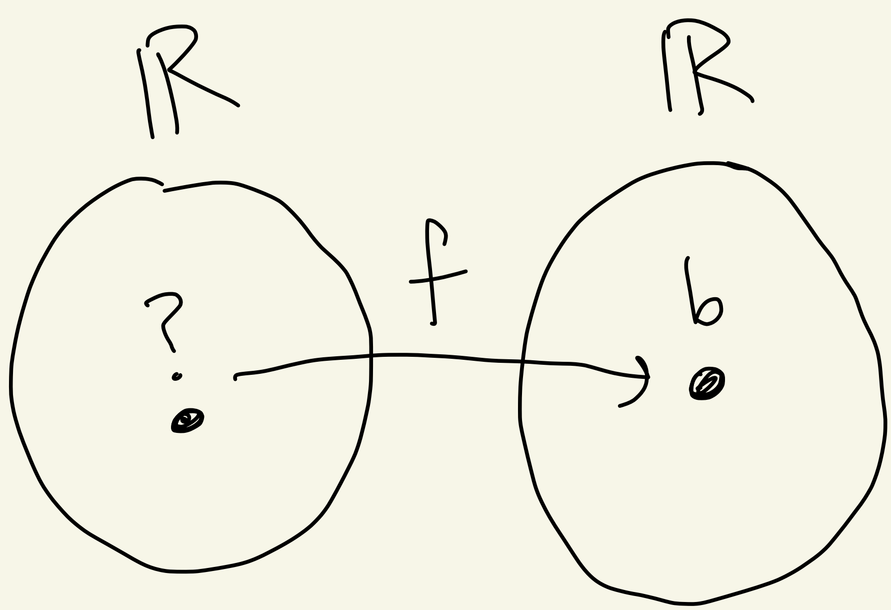
f is onto B if for every $b \in B$, there exists $a \in A$ with $f(a) = b$.



Ex: Let $f: \mathbb{R} \to \mathbb{R}$ be defined by $f(x) = -4x + 5$.

Let's show that f is onto $\mathbb{R}$.

proof:



Let $b \in \mathbb{R}$.

We must

find $a \in \mathbb{R}$, where $f(a) = b$.

Let $a = \dfrac{b-5}{-4}$.

Note $a \in \mathbb{R}$ and

$$f(a) = f\left(\frac{b-5}{-4}\right) = -4\left(\frac{b-5}{-4}\right) + 5$$

$$= (b-5) + 5 = b.$$

Thus, f is onto $\mathbb{R}$. ▨ —

Scratchwork

$$f(a) = b$$

$$-4a + 5 = b$$

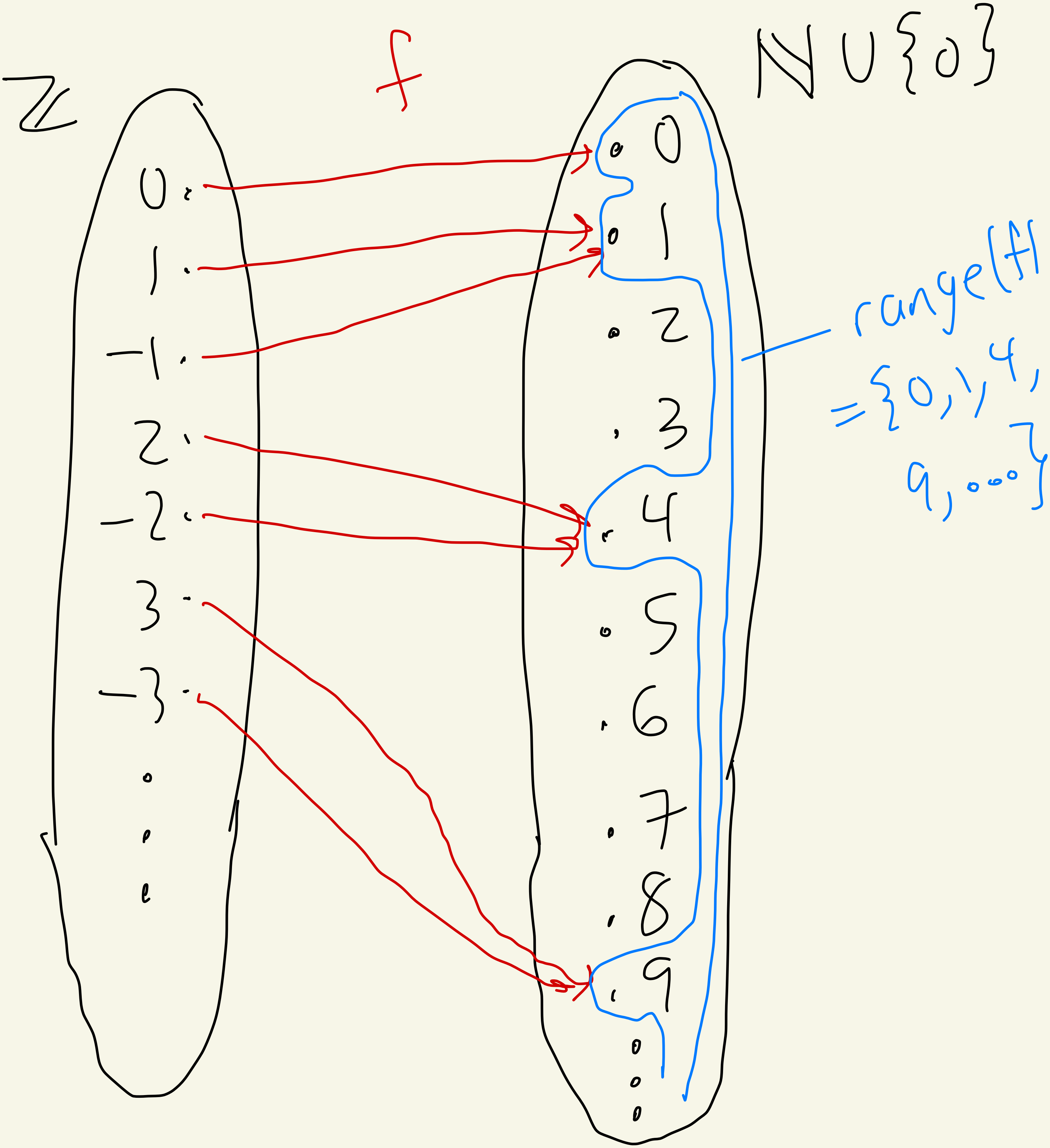
$$a = \frac{b-5}{-4}$$

How to show $f: A \to B$ is <u>not</u> onto

Find some $b \in B$ where is no $a \in A$ with $f(a) = b$

$\underline{\text{Ex:}}$ $f: \mathbb{Z} \to \mathbb{N} \cup \{0\}$

$f(x) = x^2$



f is not onto:

proof: Let $b = 2$.

Then, $b \in \mathbb{N} \cup \{0\}$.

But is no $a \in \mathbb{Z}$ with

$f(a) = 2$.

Why?

If so, then $a^2 = 2$.

Then, $a = \pm\sqrt{2} \notin \mathbb{Z}$.

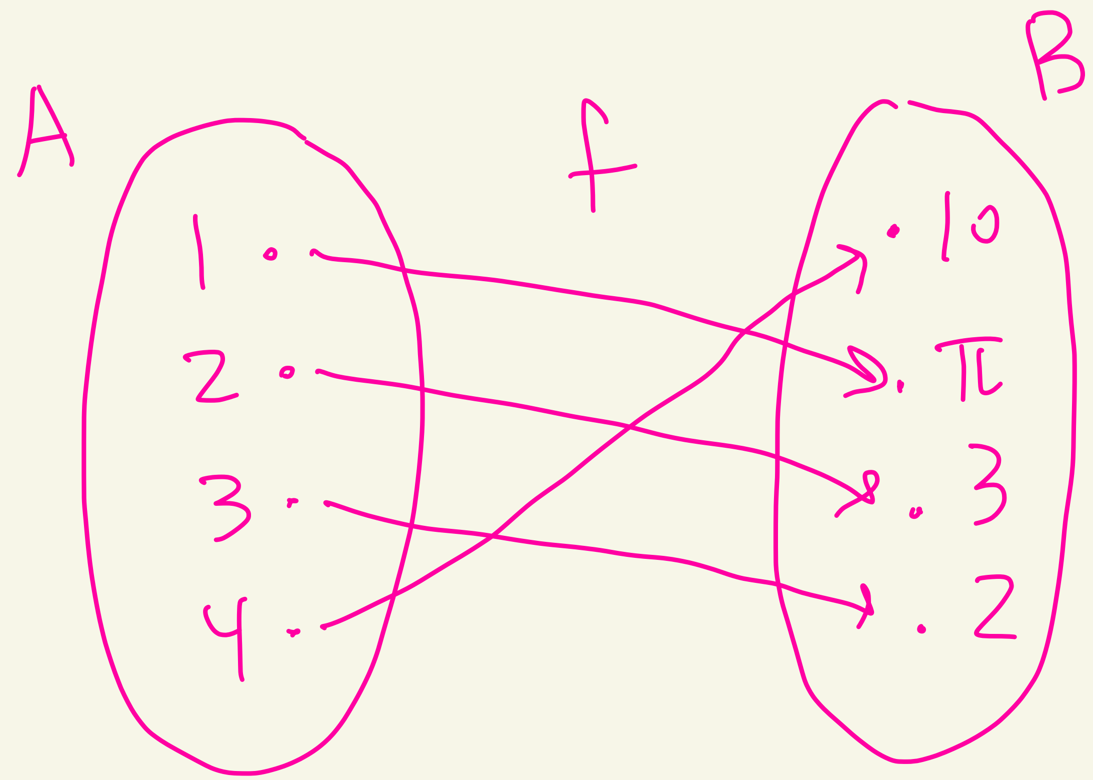
Thus, f is not onto

because $2 \notin \text{range}(f)$.

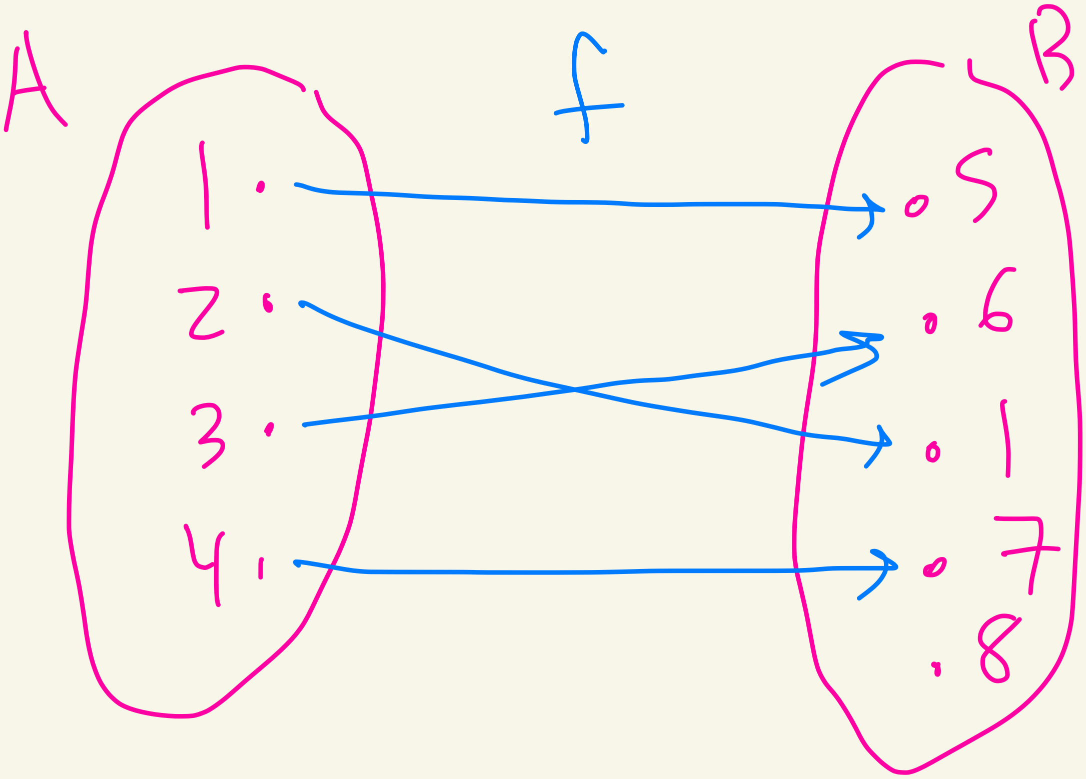
$\blacksquare$

Def: Let A and B be sets and $f: A \to B$.
We say that f is a **bijection** if f is one-to-one and onto B.

Ex:



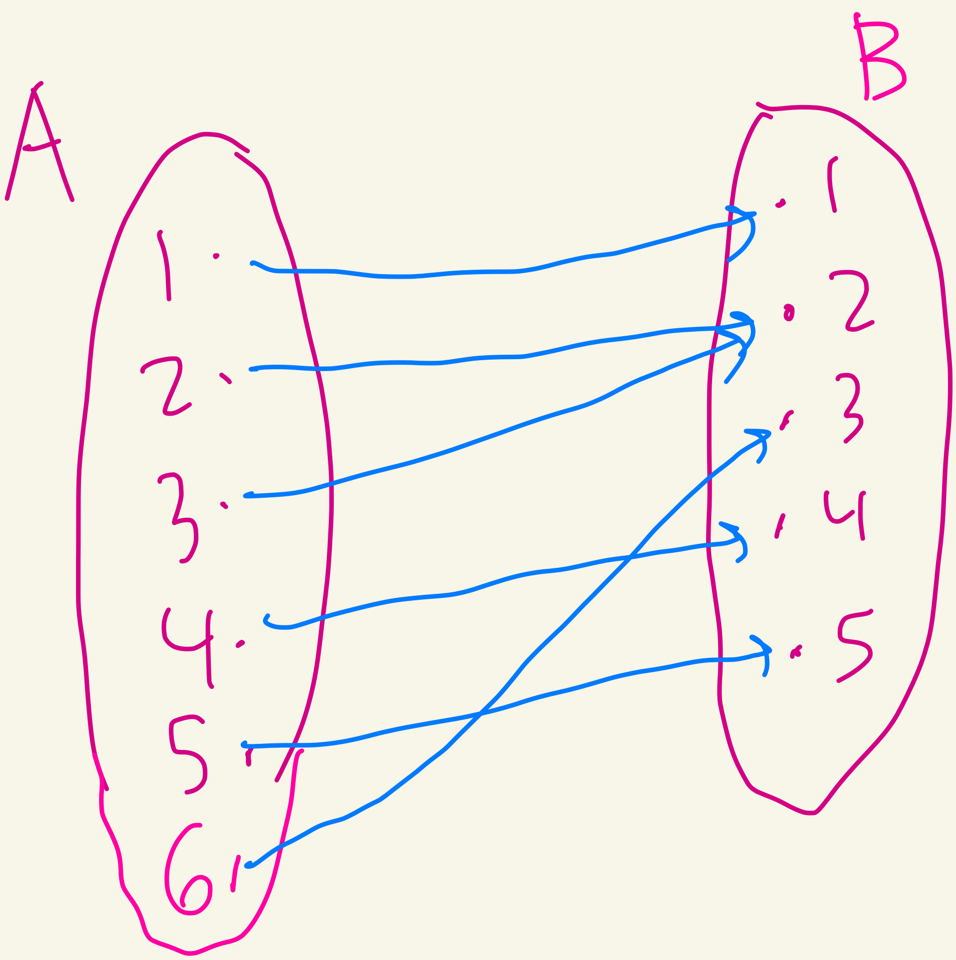
1-1 ✓
onto ✓
f is a bijection



1-1 ✓

onto B ✗

f is not a bijection



1-1 ✗

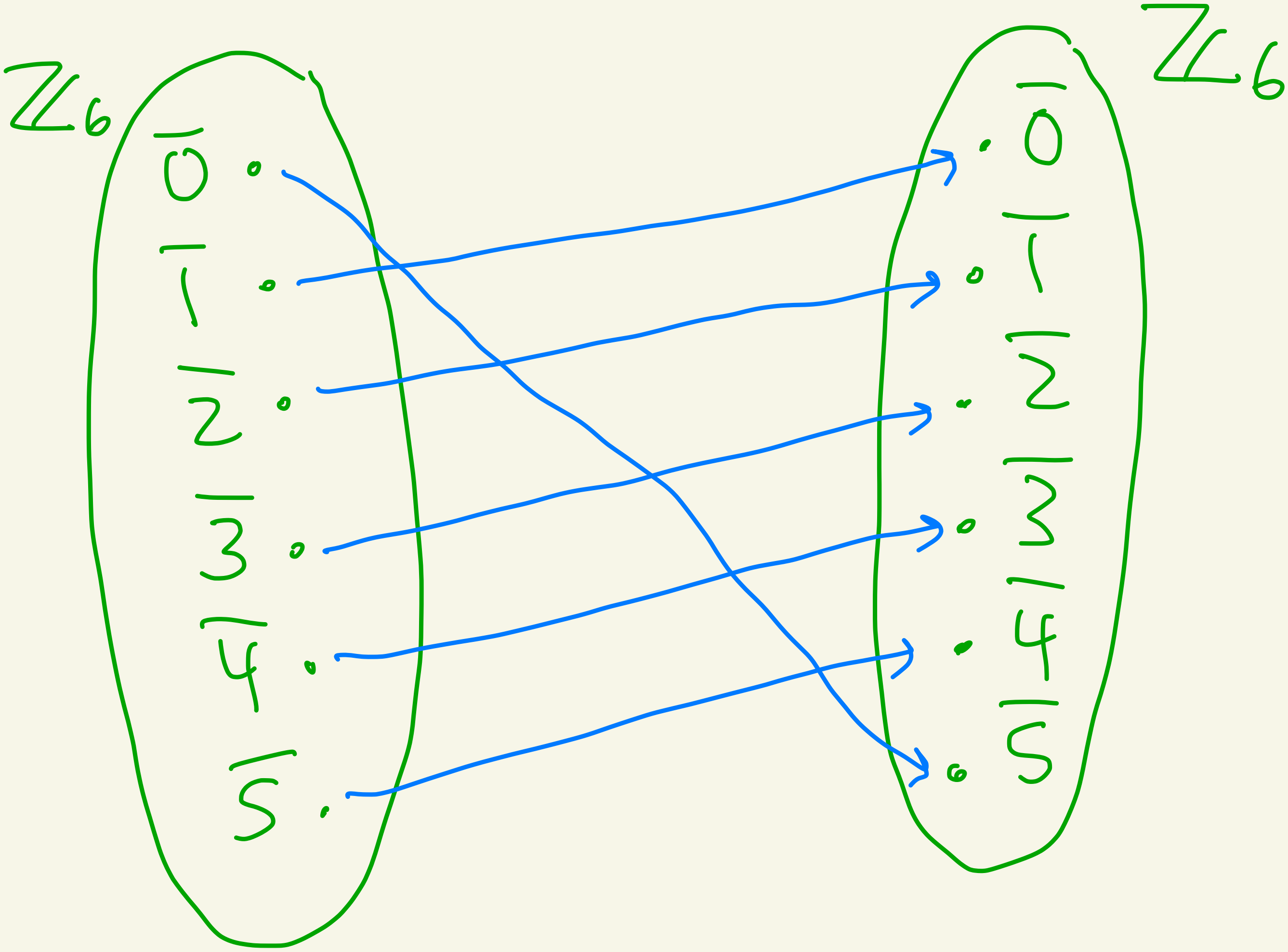
onto B ✓

f is not a bijection

<u>Ex</u>: (from Homework)

Given $a \in \mathbb{Z}$, define $g_a : \mathbb{Z}_n \to \mathbb{Z}_n$ by $g_a(\overline{x}) = \overline{x} + \overline{a}$

$\boxed{\text{Ex:}}$ $g_5 : \mathbb{Z}_6 \to \mathbb{Z}_6$, $g_5(\overline{x}) = \overline{x} + \overline{5}$



$g_5(\overline{0}) = \overline{0} + \overline{5} = \overline{5}$

$g_5(\overline{1}) = \overline{1} + \overline{5} = \overline{6} = \overline{0}$

$g_5(\overline{2}) = \overline{5} + \overline{2} = \overline{7} = \overline{1}$

$\vdots$

In the HW you show g_a is well-defined.

$\boxed{\text{Claim: Given } a \in \mathbb{Z}, \text{ the function } g_a : \mathbb{Z}_n \to \mathbb{Z}_n \text{ given by } g_a(\overline{x}) = \overline{x} + \overline{a} \text{ is a bijection}}$

proof:

(one-to-one)

Suppose $g_a(\overline{x_1}) = g_a(\overline{x_2})$ where $\overline{x_1}, \overline{x_2} \in \mathbb{Z}_n$.

Then, $\overline{x_1} + \overline{a} = \overline{x_2} + \overline{a}$.

Then, $(\overline{x_1} + \overline{a}) + \overline{-a} = (\overline{x_2} + \overline{a}) + \overline{-a}$.

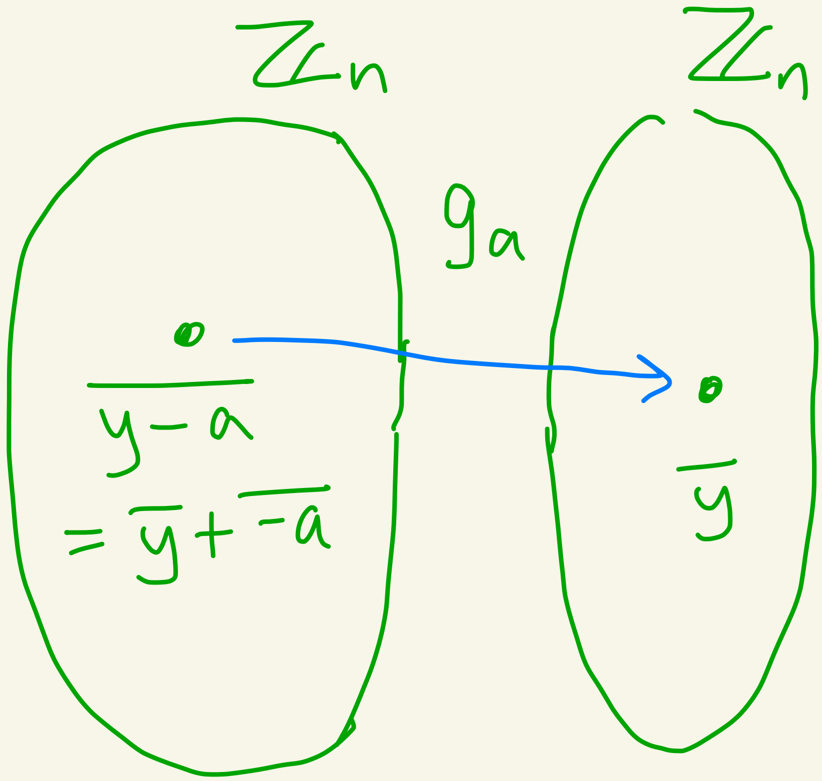
Thus, $\overline{x_1} + \overline{0} = \overline{x_2} + \overline{0}$.

So, $\overline{x_1} = \overline{x_2}$.

Thus, g_a is one-to-one.

(onto)

Let $\overline{y} \in \mathbb{Z}_n$, where $y \in \mathbb{Z}$.



Then,

$$\overline{y-a} \in \mathbb{Z}_n$$

because $y - a \in \mathbb{Z}$.

And,

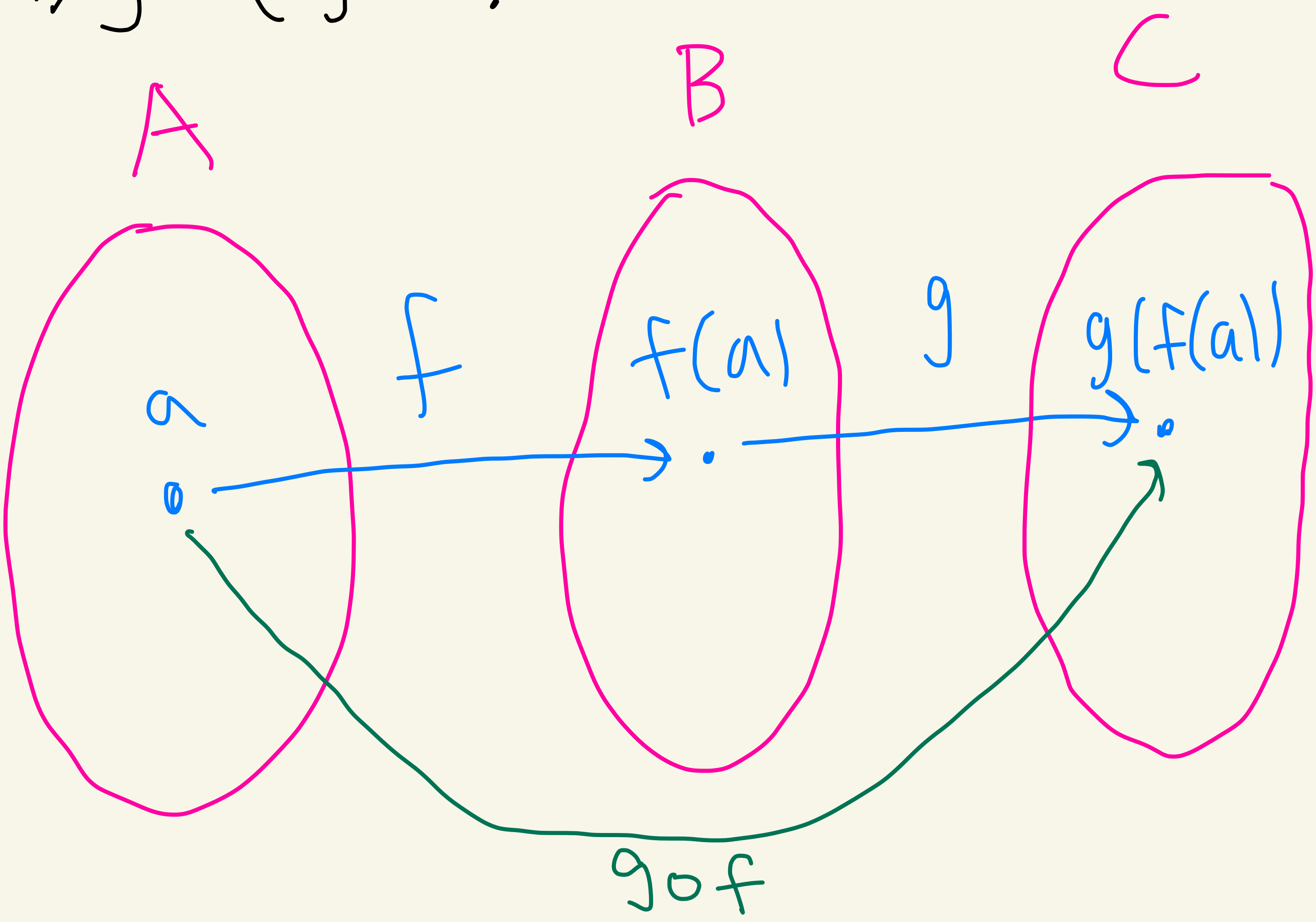
$$g_a(\overline{y-a}) = \overline{y-a} + \overline{a}$$

$$= \overline{y-a+a}$$

$$= \overline{y}$$

So, g_a is onto. $\blacksquare$

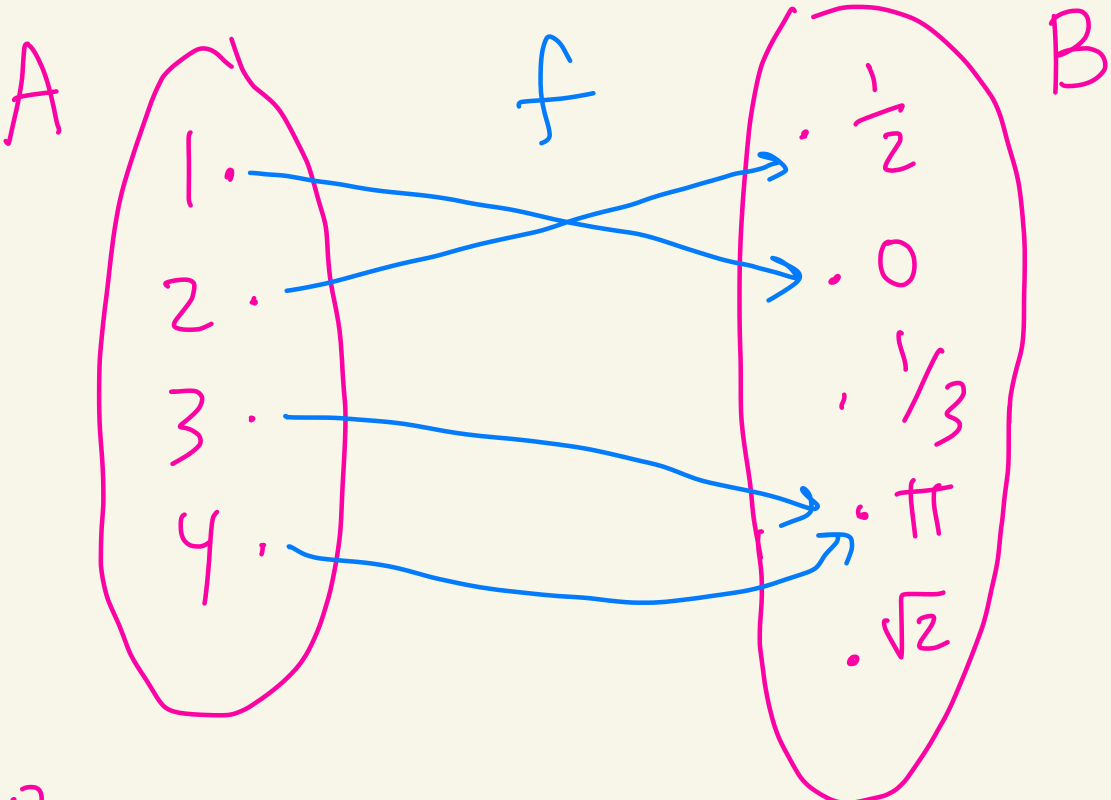
Def: Let A, B, C be sets. Let $f: A \to B$ and $g: B \to C$. Define the composition of f and g to be the function $(g \circ f): A \to C$ by $(g \circ f)(a) = g(f(a))$

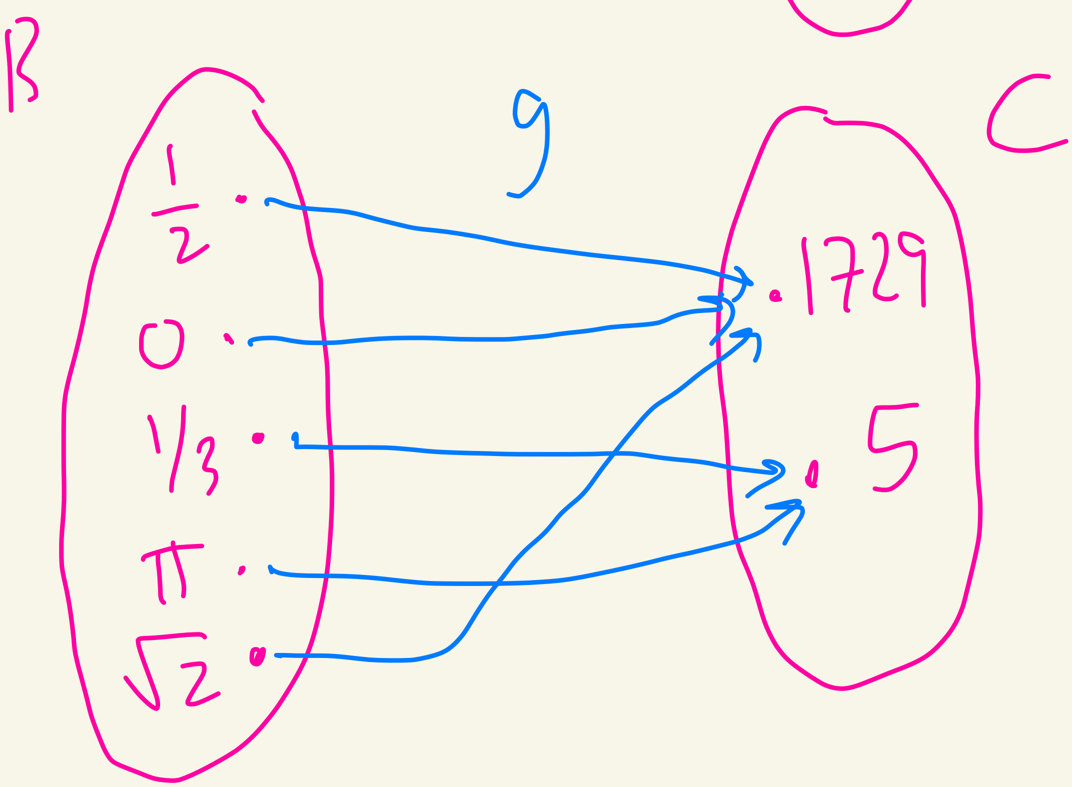


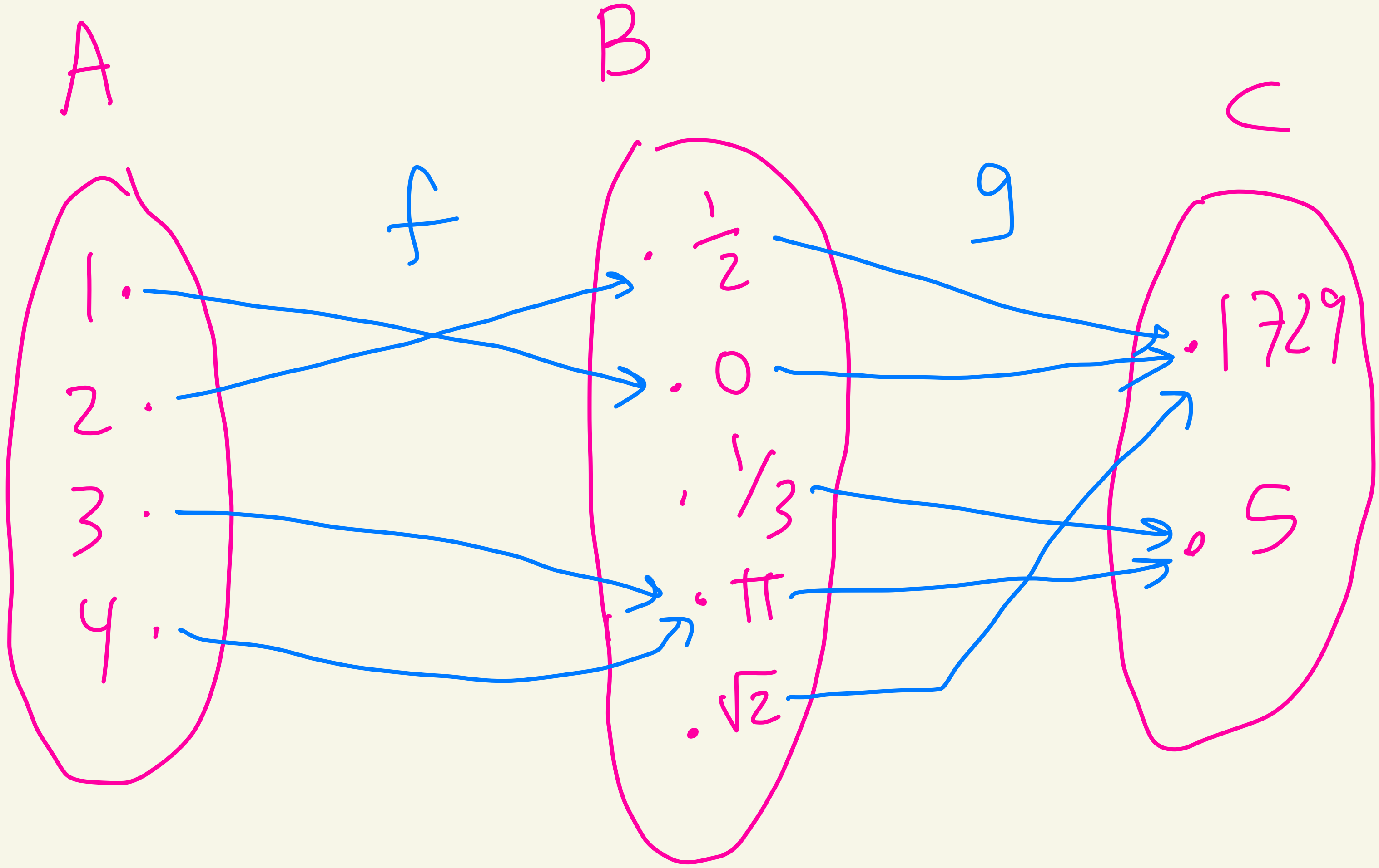
Ex: $A = \{1, 2, 3, 4\}$

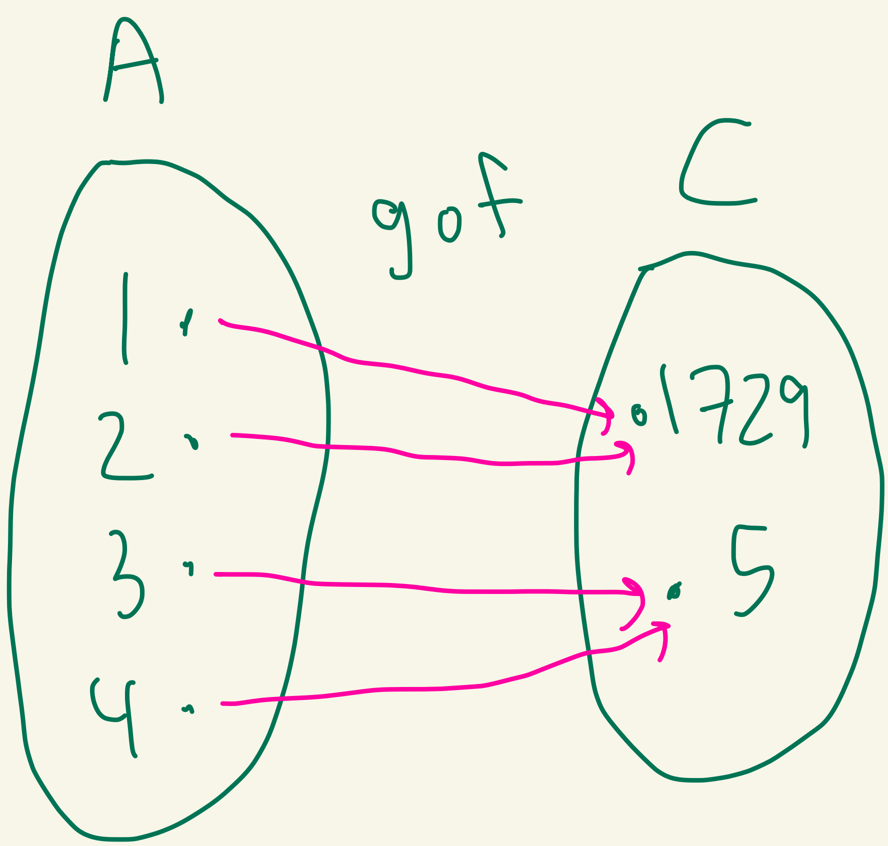
$B = \{\frac{1}{2}, 0, \frac{1}{3}, \pi, \sqrt{2}\}$

$C = \{1729, 5\}$









$$(g \circ f)(1) = g(f(1))$$
$$= g(0)$$
$$= 1729$$

$$(g \circ f)(2) = g(f(2))$$
$$= g\left(\tfrac{1}{2}\right)$$
$$= 1729$$

Note: $g \circ f$ is onto but not one-to-one

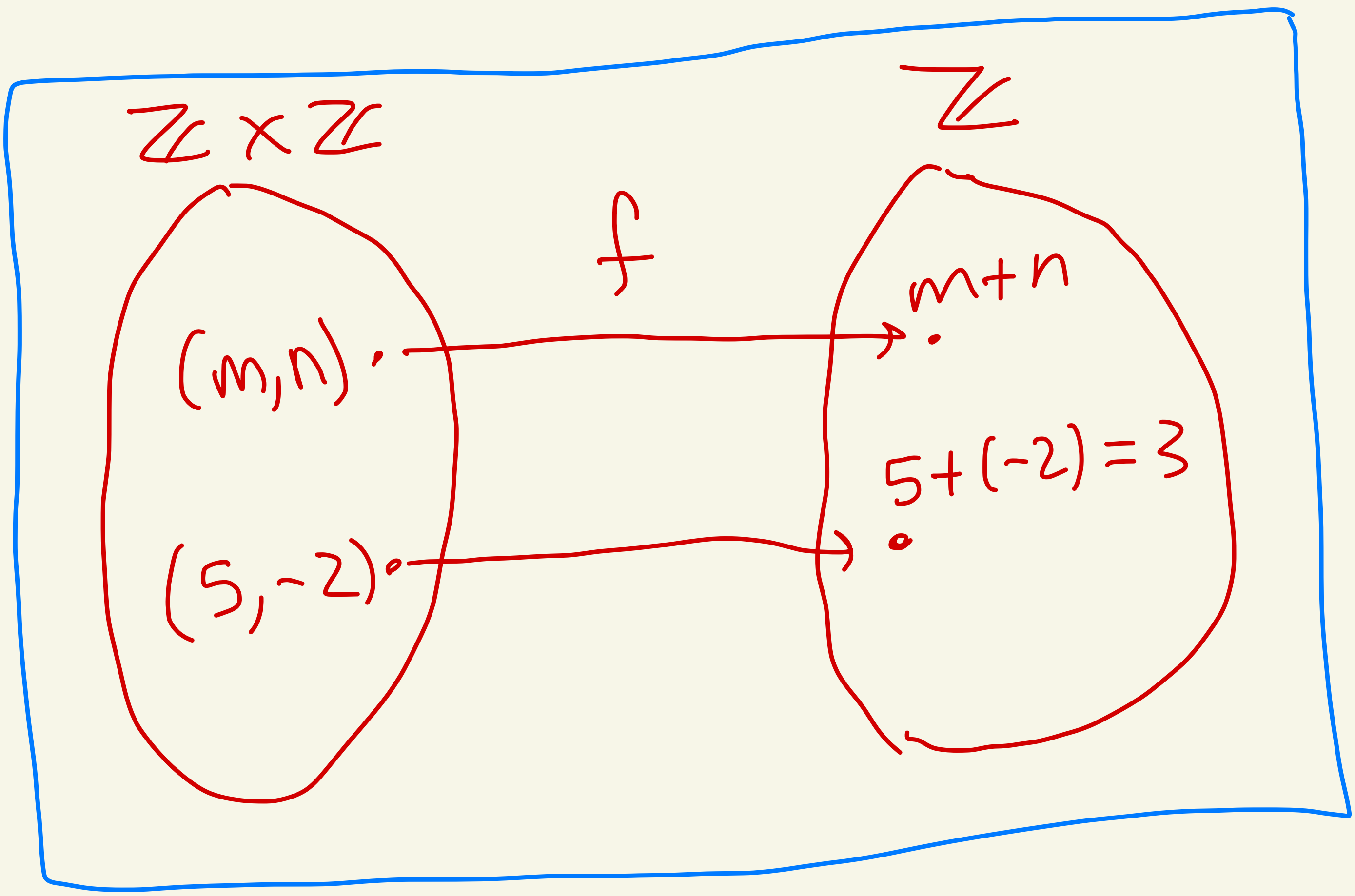
Ex: (Hammock 12.4 #9)

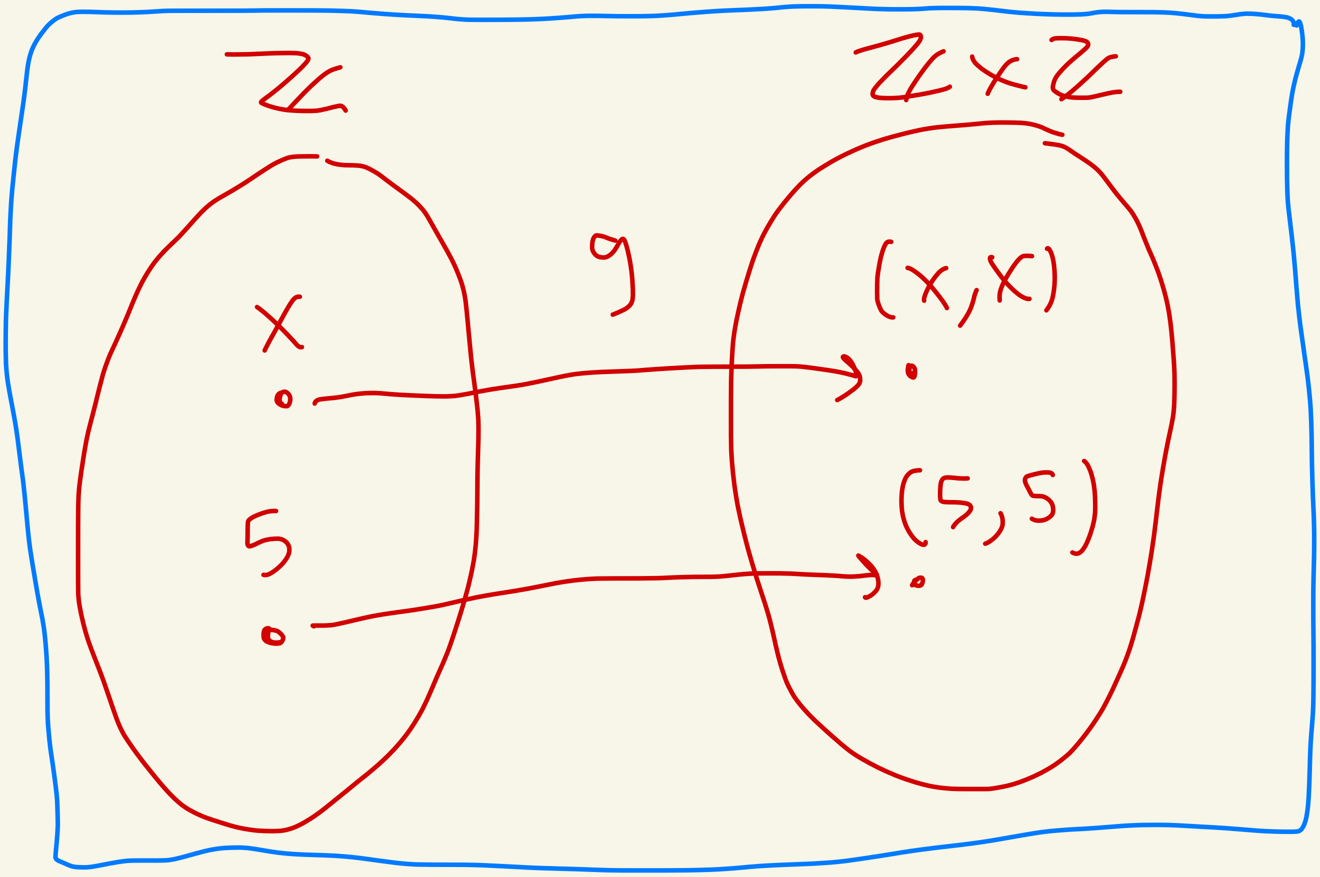
Define $f: \mathbb{Z} \times \mathbb{Z} \to \mathbb{Z}$ where

$$f(m,n) = m+n$$

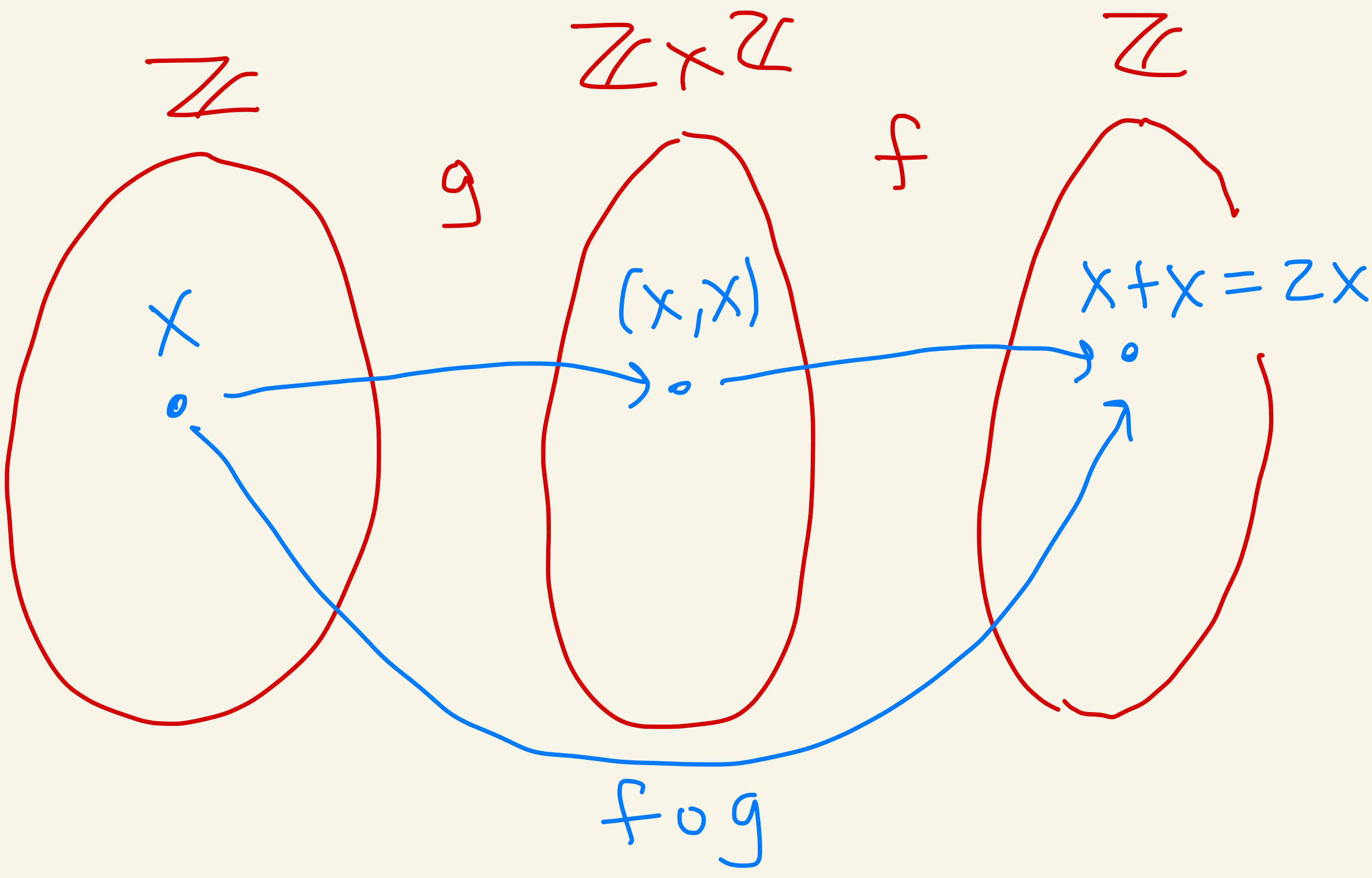
and $g: \mathbb{Z} \to \mathbb{Z} \times \mathbb{Z}$ where

$$g(x) = (x,x)$$





Find formulas for $f \circ g$ and $g \circ f$

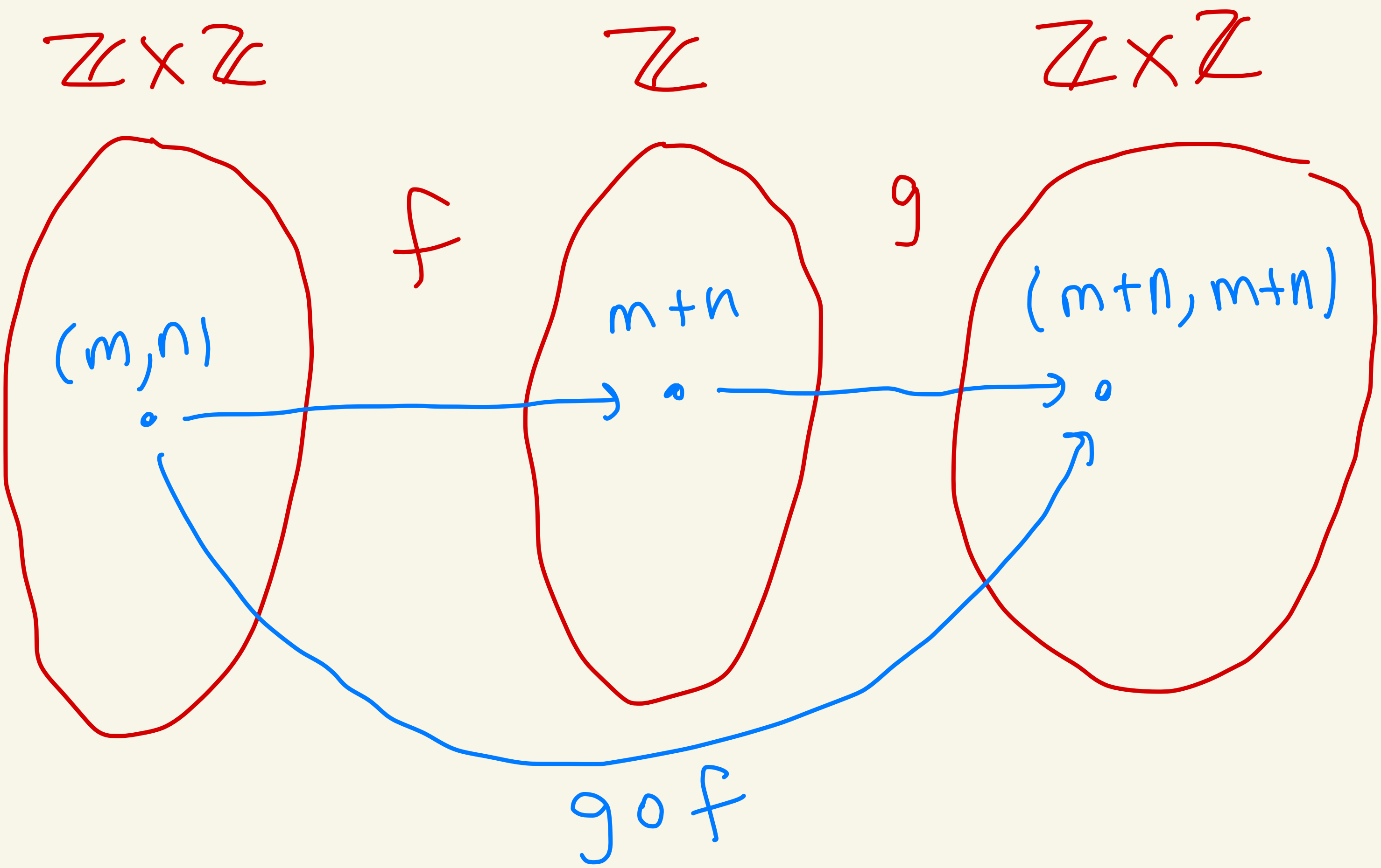


$f \circ g : \mathbb{Z} \to \mathbb{Z}$

$(f \circ g)(x) = f(g(x))$

$= f(x, x) = x + x = 2x$

So, $(f \circ g)(x) = 2x$



$g \circ f : \mathbb{Z} \times \mathbb{Z} \to \mathbb{Z} \times \mathbb{Z}$

$(g \circ f)(m, n) = g(f(m, n))$

$$= g(m+n)$$
$$= (m+n, m+n)$$

So, $(g \circ f)(m,n) = (m+n, m+n)$

Question: Is g 1-1?
Is g onto?

Claim: g is 1-1

pf: Suppose $g(x_1) = g(x_2)$ where $x_1, x_2 \in \mathbb{Z}$.

$g(x) = (x,x)$

Then, $(x_1, x_1) = (x_2, x_2)$

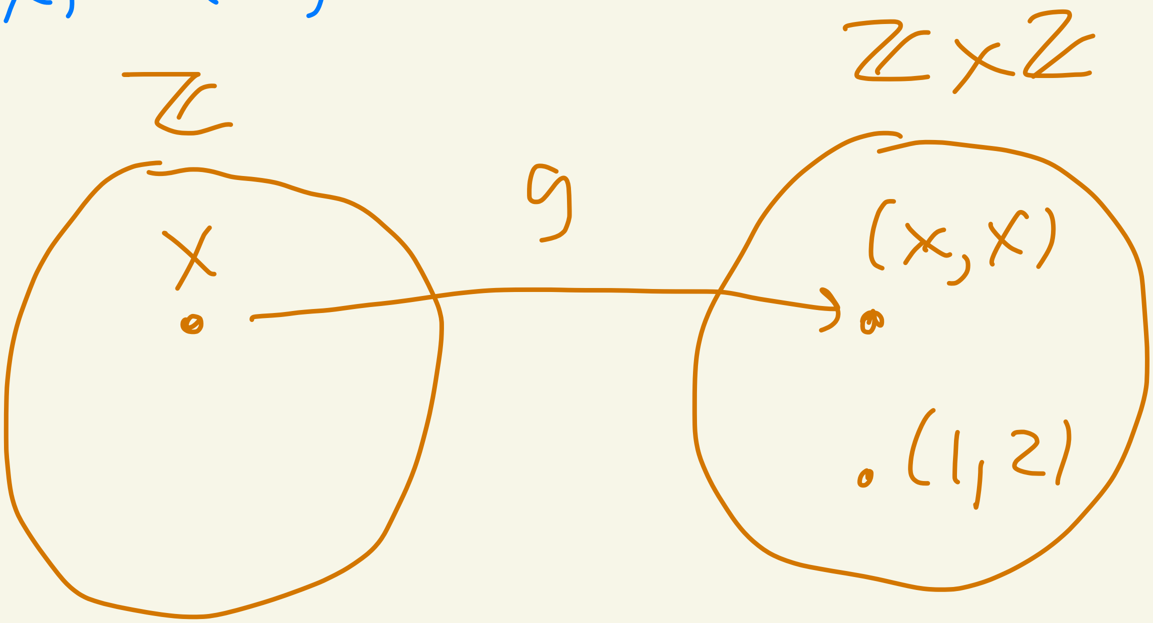
So, $x_1 = x_2$. $\blacksquare$

Claim: g is not onto.

Pf: Let $(1,2) \in \mathbb{Z} \times \mathbb{Z}$.

There is no $x \in \mathbb{Z}$ with

$$g(x) = (x,x) = (1,2).$$



So, $(1,2) \notin \text{range}(g)$

So, g is not onto.

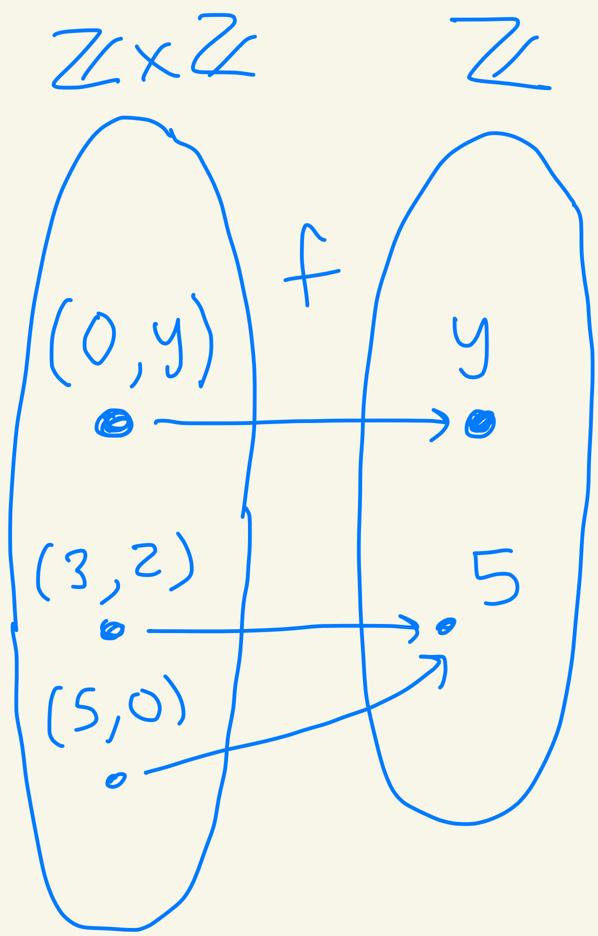
Recall that

$f : \mathbb{Z} \times \mathbb{Z} \to \mathbb{Z}$ where $f(m,n) = m+n$.

Question: Is f onto?

Is f 1-1?

Claim: f is onto



proof:

Let $y \in \mathbb{Z}$.

Then, $(0,y) \in \mathbb{Z} \times \mathbb{Z}$

and $f(0,y) = 0 + y$

$= y$.

$\blacksquare$

Claim: f is not 1-1

proof: $f(3,2) = 5 = f(5,0)$

but $(3,2) \neq (5,0)$.

See picture above. $\blacksquare$

Theorem: Let A, B, C be sets and $f: A \to B$ and $g: B \to C$.

① If f and g are both onto, then $g \circ f$ is onto.

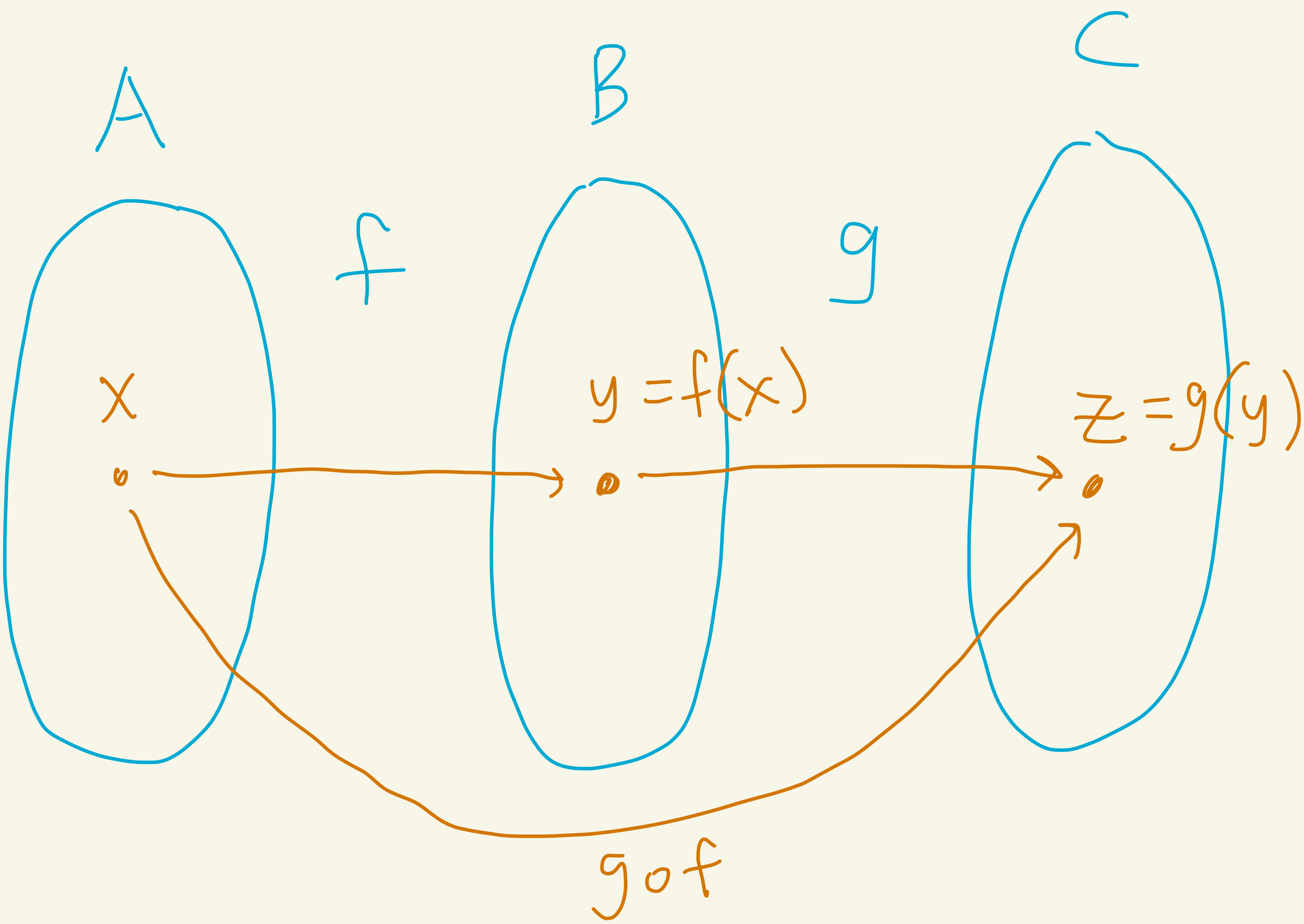
② If f and g are both 1-1, then $g \circ f$ is 1-1.

(3) If f and g are both bijections (1-1 and onto), then $g \circ f$ is a bijection.

proof:

(1) Suppose f and g are both onto.

Note $g \circ f : A \to C$.



Let $z \in C$.

Since g is onto C, there exists $y \in B$ where $g(y) = z$.

Since f is onto B, there exists $x \in A$ where $f(x) = y$.

Then $(g \circ f)(x) = g(f(x))$

$$= g(y) = z.$$

So, $g \circ f$ is onto [because there exists $x \in A$ with $(g \circ f)(x) = z$]

2 Suppose f and g are both 1-1.

Suppose $(g \circ f)(x_1) = (g \circ f)(x_2)$
where $x_1, x_2 \in A$.

Then, $g(f(x_1)) = g(f(x_2))$.

Since g is 1-1 and
$g(f(x_1)) = g(f(x_2))$,
this implies that $f(x_1) = f(x_2)$.

Since f is 1-1 and
$f(x_1) = f(x_2)$ this
implies that $x_1 = x_2$.

So, $(g \circ f)(x_1) = (g \circ f)(x_2)$.
implies that $x_1 = x_2$.

Thus, $g \circ f$ is 1-1.

③ Suppose f and g are both bijections (1-1 and onto).

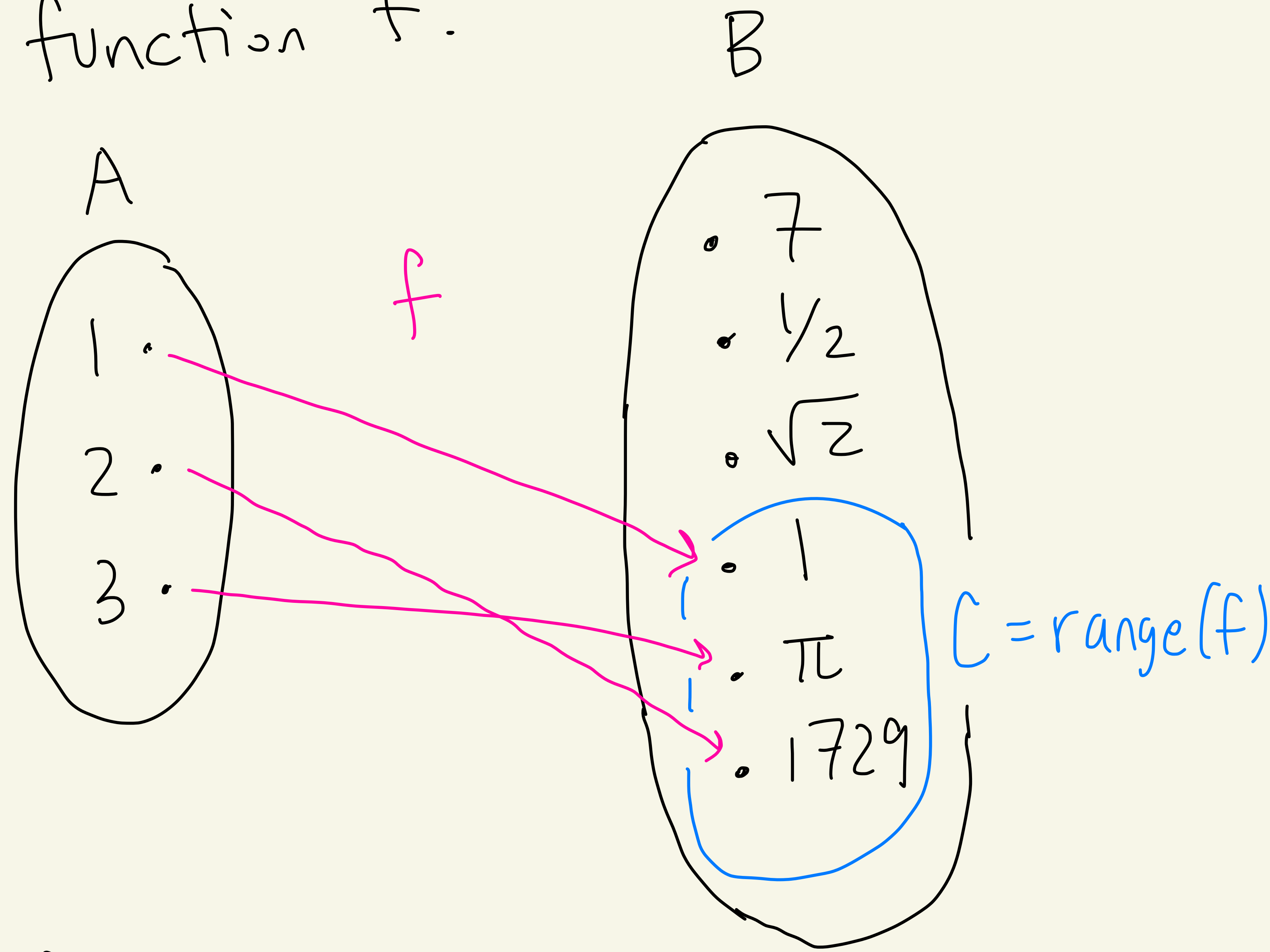
By 1, this implies that
$g \circ f$ will be onto.

By 2, this implies that
$g \circ f$ will be 1-1.

So, $g \circ f$ is a bijection. ∎

Now we talk about inverse functions.

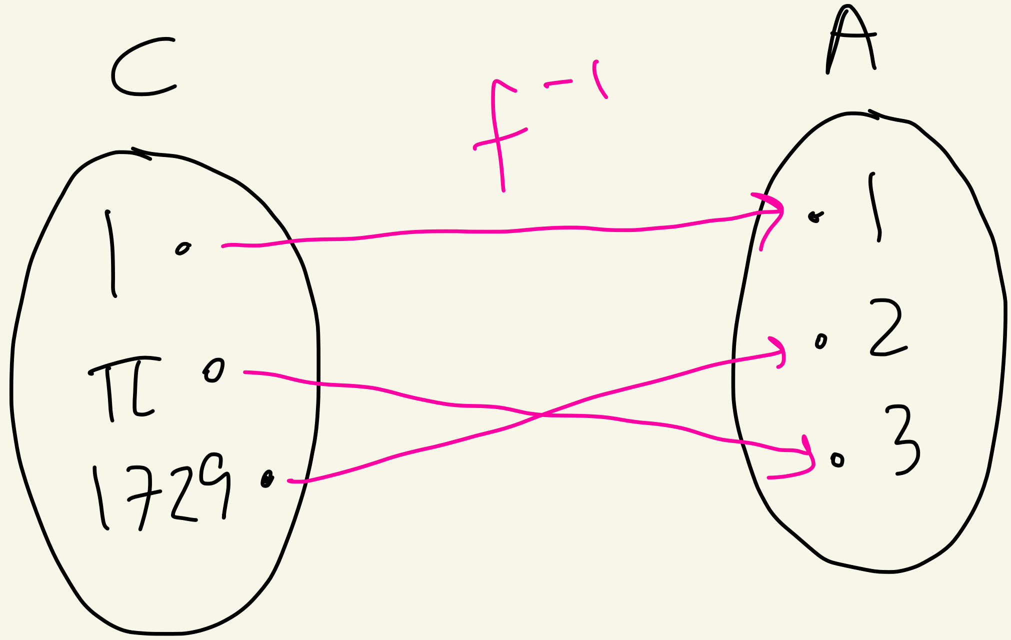
Ex: Consider the following function f.



f is one-to-one.

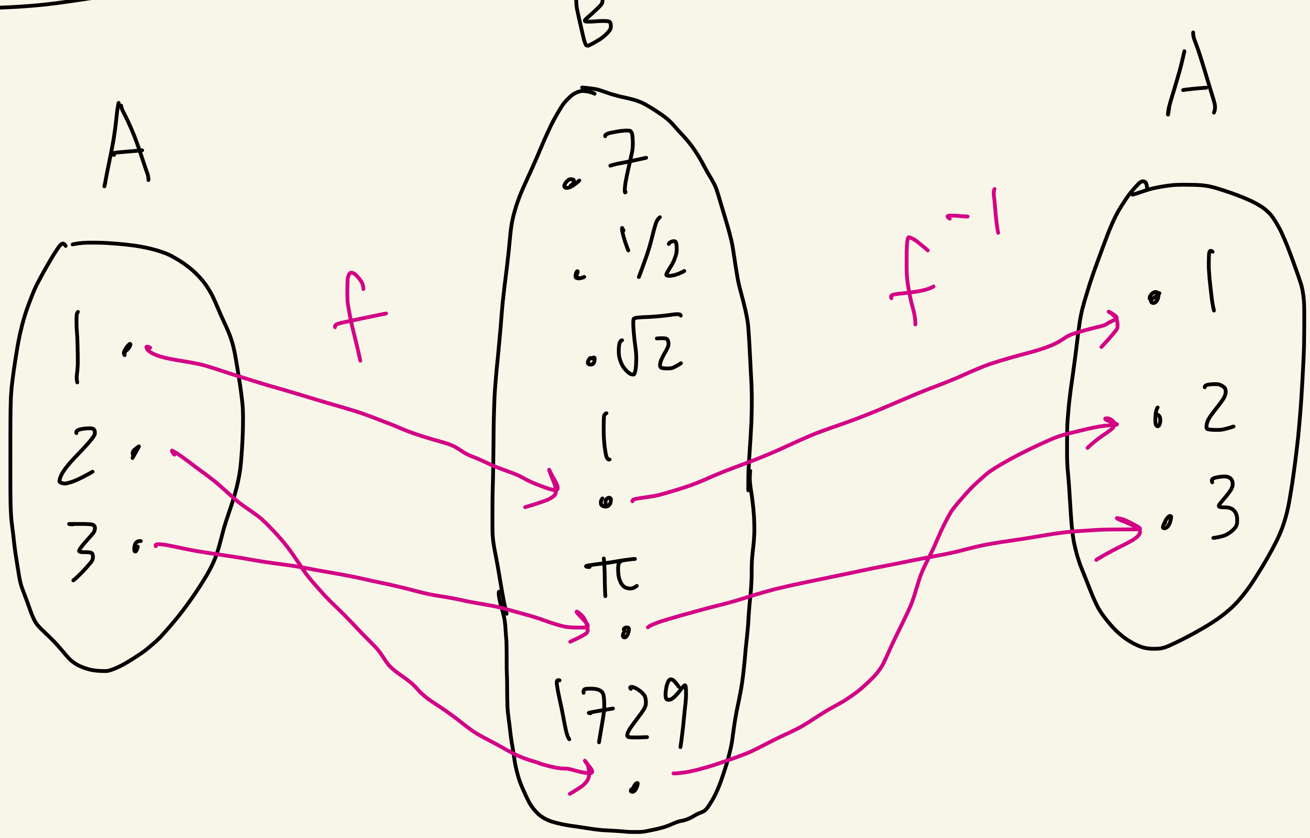
So we can create $f^{-1}: C \to A$ by reversing the arrows

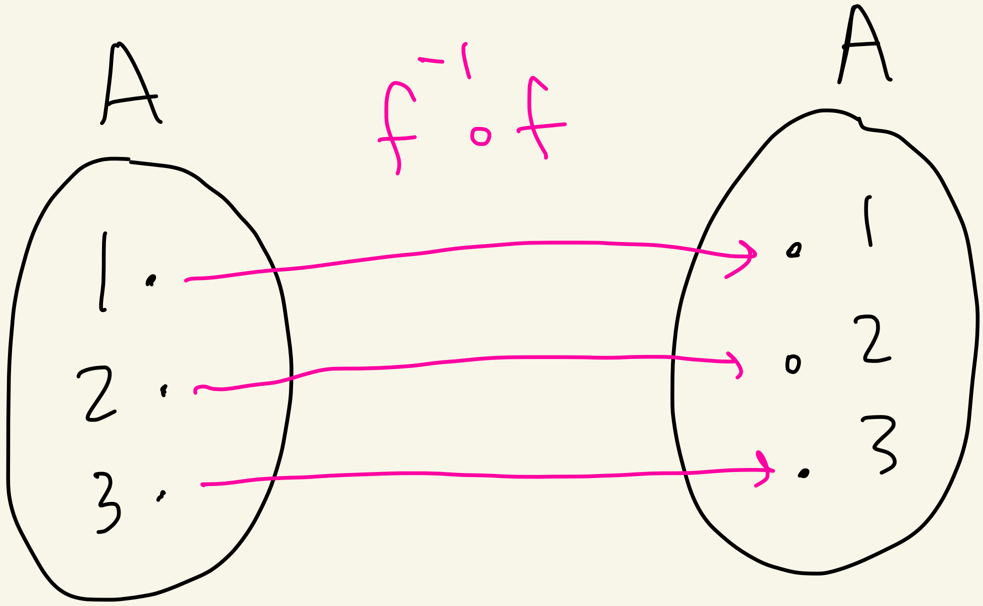
f^{-1} will be well-defined since f is 1-1 and so each element of C only has one arrow to reverse.



$$\mathrm{domain}(f^{-1}) = C = \mathrm{range}(f)$$
$$\mathrm{range}(f^{-1}) = A = \mathrm{domain}(f)$$

What is $f^{-1} \circ f$?





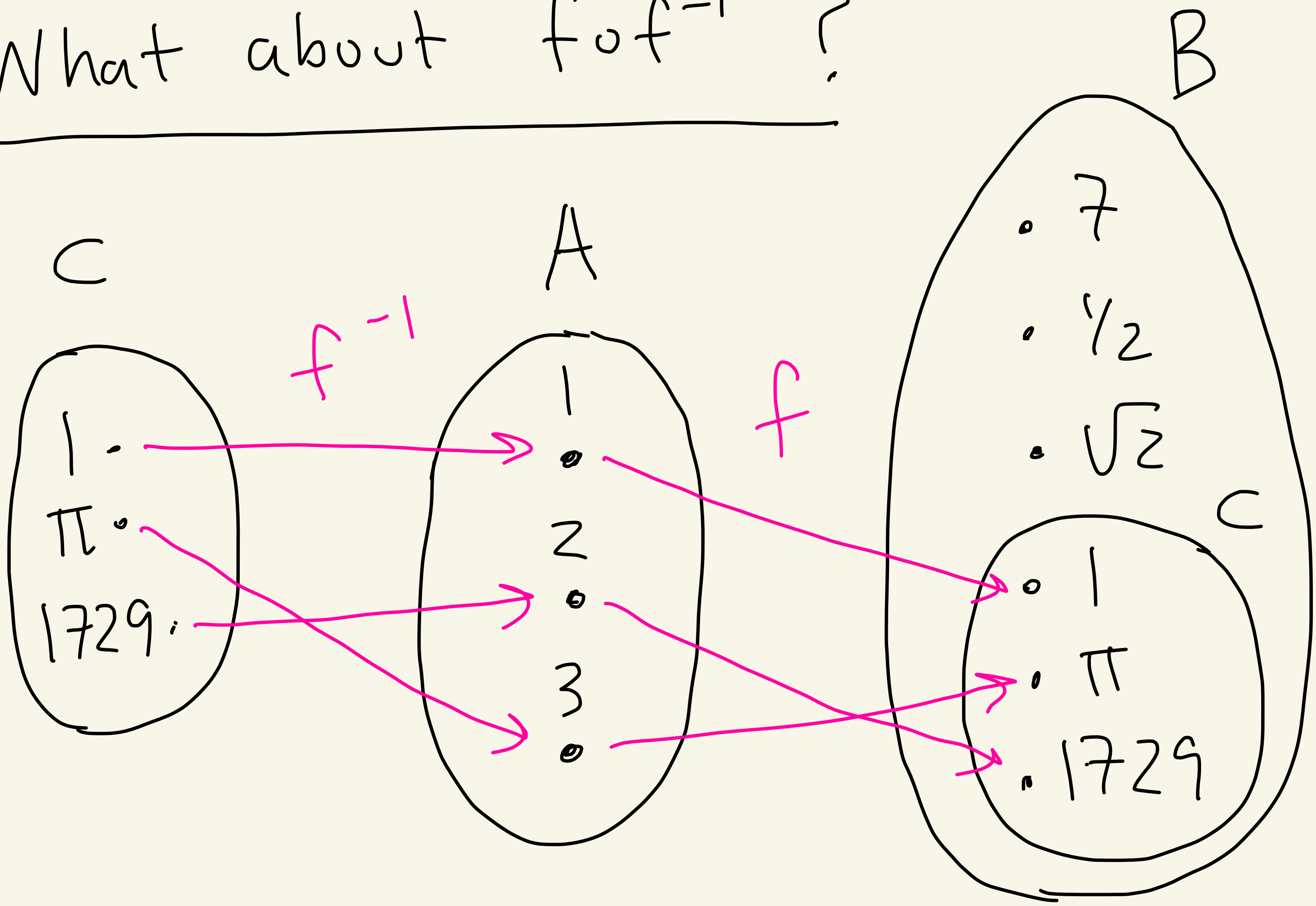
$$(f^{-1}\circ f)(1) = f^{-1}(f(1)) = f^{-1}(1) = 1$$

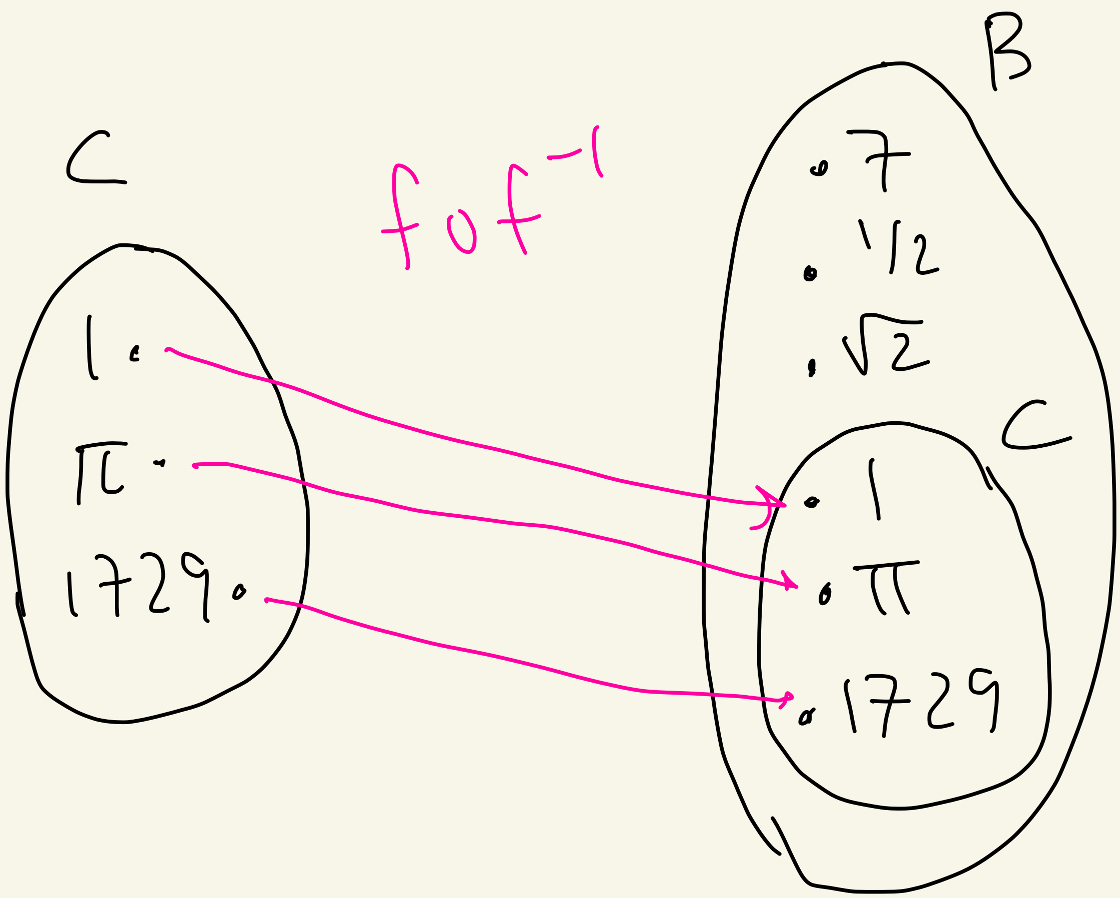
$$(f^{-1}\circ f)(2) = f^{-1}(f(2)) = f^{-1}(1729) = 2$$

$$(f^{-1}\circ f)(3) = f^{-1}(f(3)) = f^{-1}(\pi) = 3$$

Thus, $f^{-1}\circ f = i_A$ (the identity function on A)

What about $f \circ f^{-1}$?





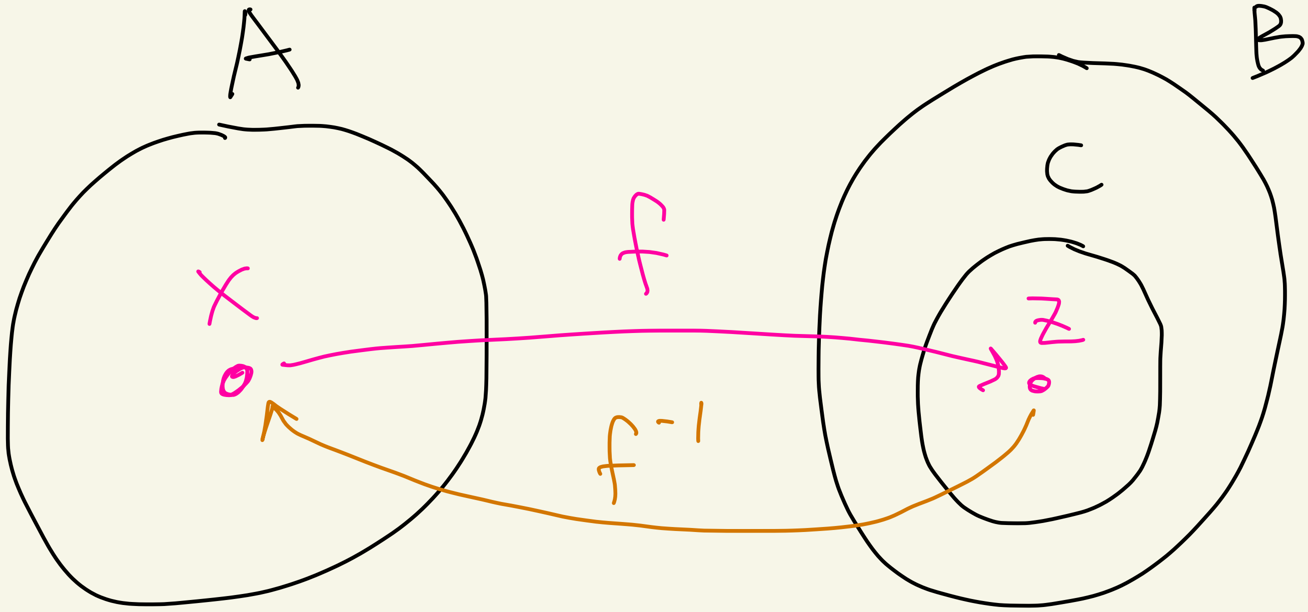
We see that

$$(f \circ f^{-1})(z) = z = i_C(z)$$

for all $z \in C$.

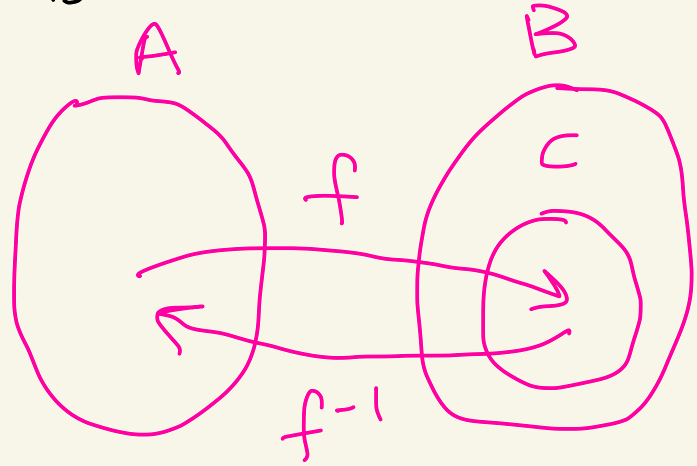
Def: Let A and B be sets. Let $f: A \to B$ be a one-to-one function. Let $C = \text{range}(f)$.

Define the *inverse function of f* to be $f^{-1}: C \to A$ such that $f^{-1}(z) = x$ where $f(x) = z$.



Note: f^{-1} is well-defined because f is one-to-one. There is one and only one arrow to reverse for each z in C.

<u>Theorem:</u> Let A, B be sets. Let $f: A \to B$ be a one-to-one function. Let $C = \text{range}(f)$. Let $f^{-1}: C \to A$ be the inverse of f. Then:



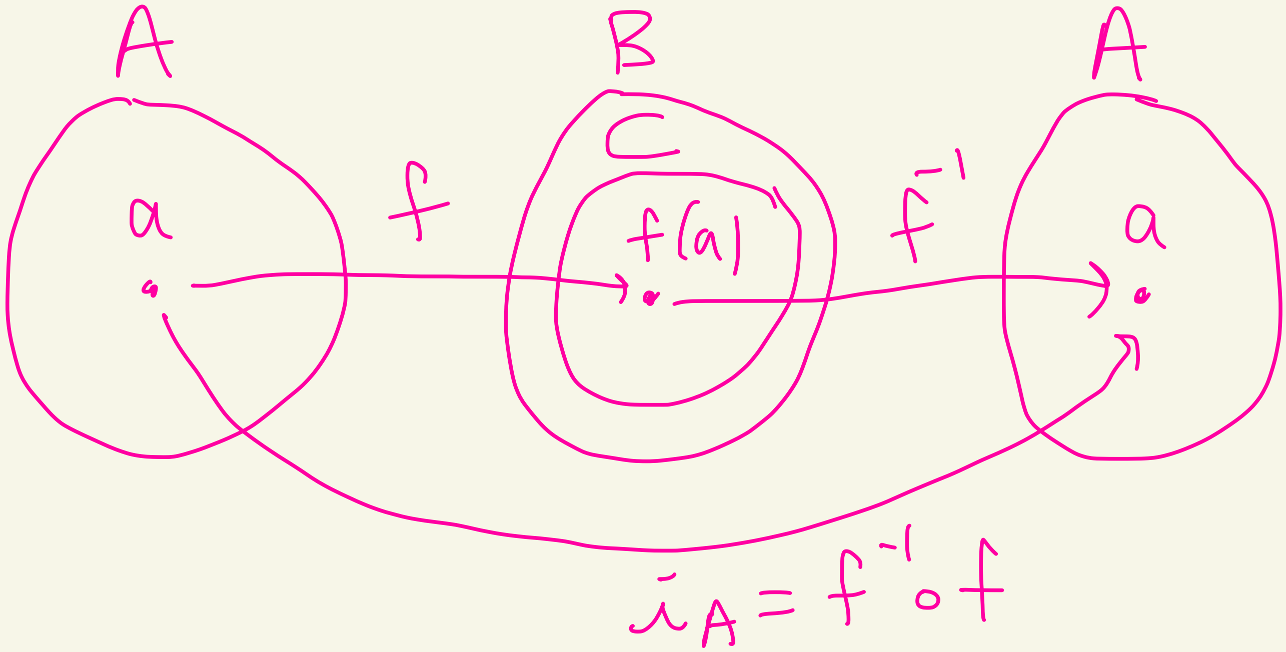
① $\text{domain}(f^{-1}) = \text{range}(f) = C$

② $\text{range}(f^{-1}) = \text{domain}(f) = A$
In particular, f^{-1} is onto A.

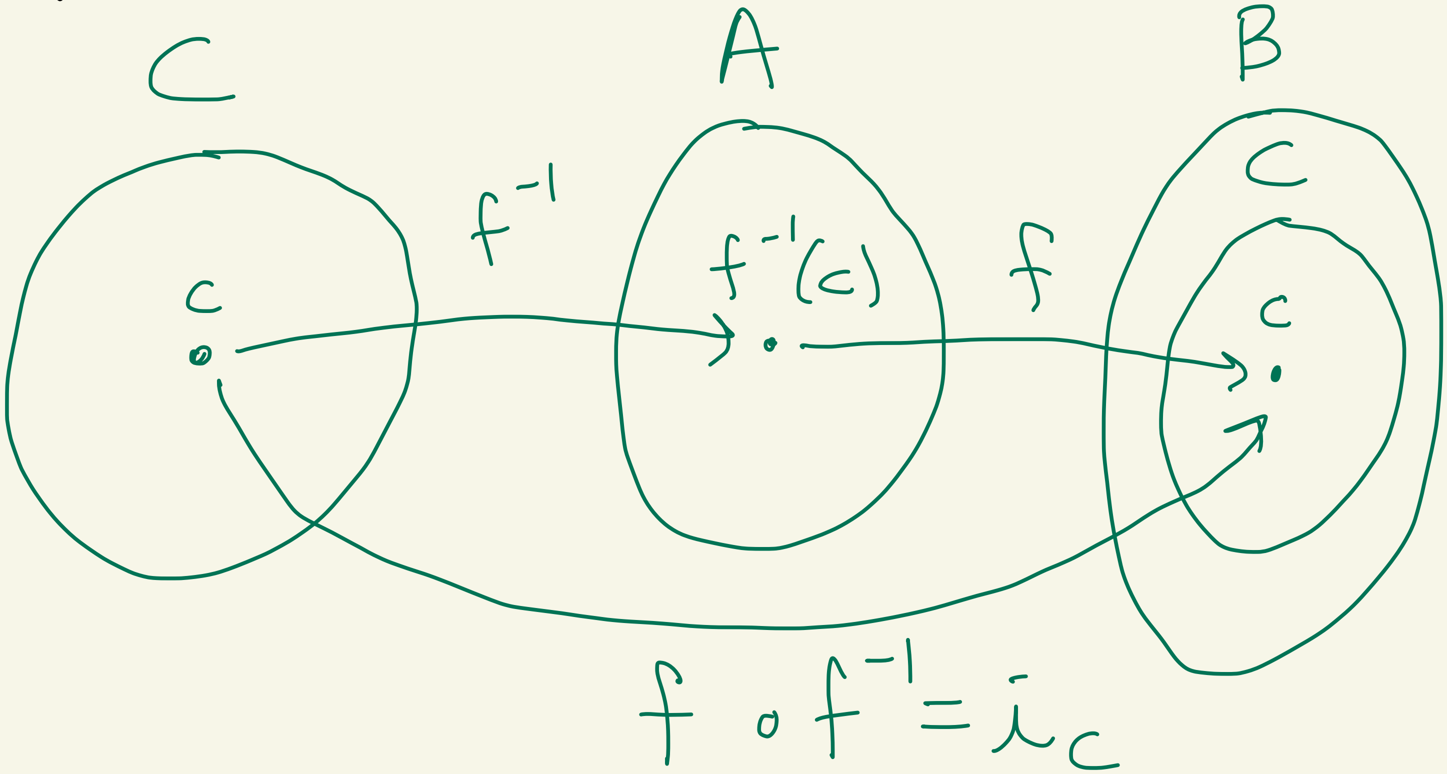
③ f^{-1} is one-to-one

④ $(f^{-1} \circ f)(a) = a$ for all $a \in A$.

So, $f^{-1} \circ f = i_A$



⑤ $(f \circ f^{-1})(c) = c$ for all $c \in C$.



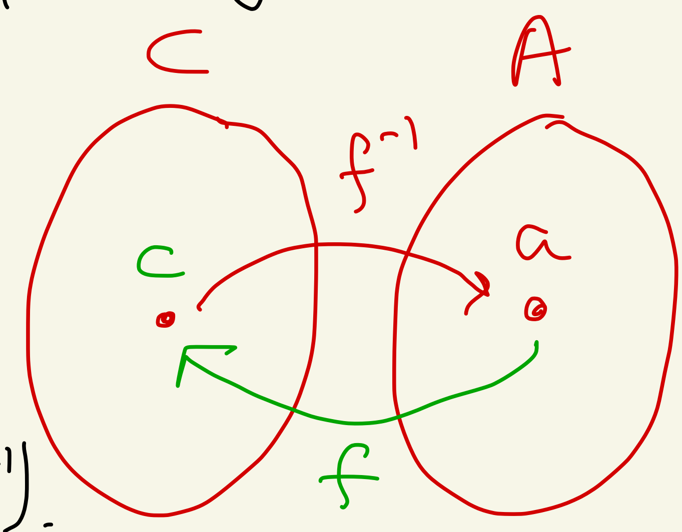
⑥ If $g: C \to A$ and $g \circ f = i_A$, then $g = f^{-1}$.

[⑥ Is a way to check that $g = f^{-1}$]

proof:

① By def of f^{-1} we have
$\text{domain}(f^{-1}) = C = \text{range}(f)$.

② Let's show that $\text{range}(f^{-1}) = A$.



By def of f^{-1} we know $\text{range}(f^{-1}) \subseteq A$.

Why is $A \subseteq \text{range}(f^{-1})$.

Let $a \in A$.

Let $c = f(a)$

And, $f^{-1}(c) = a$ by def of f^{-1}.

So, $a \in \text{range}(f^{-1})$.

Thus, $A \subseteq \text{range}(f^{-1})$

Therefore, $A = \text{range}(f^{-1})$.

③ Let's show that f^{-1} is one-to-one.

Suppose $f^{-1}(c_1) = f^{-1}(c_2)$ where $c_1, c_2 \in C$.

We need to show that $c_1 = c_2$.

Let $a = f^{-1}(c_1) = f^{-1}(c_2)$.

Since $a = f^{-1}(c_1)$ we know that $f(a) = c_1$.

Since $a = f^{-1}(c_2)$ we know that $f(a) = c_2$.

So, $c_1 = f(a) = c_2$.

Thus, f^{-1} is one-to-one.

(4) Let's show that $f^{-1} \circ f = i_A$.

Let $a \in A$.

Set $c = f(a)$.

So, $f^{-1}(c) = a$ by def of f^{-1}.

Then,

$$
\begin{aligned}
(f^{-1} \circ f)(a) &= f^{-1}(f(a)) \\
&= f^{-1}(c) \\
&= a \\
&= i_A(a)
\end{aligned}
$$

Thus, $(f^{-1} \circ f)(a) = i_A(a)$ for all $a \in A$.

So, $f^{-1} \circ f = i_A$

5. Let's show that
$(f \circ f^{-1})(c) = c$ for all $c \in C$.

Let $c \in C$.
Then, $f^{-1}(c) = a$ where $a \in A$
and $f(a) = c$.

Thus,

$$\begin{aligned}(f \circ f^{-1})(c) &= f(f^{-1}(c)) \\ &= f(a) \\ &= c \\ &= i_C(c)\end{aligned}$$

6 Let $g: C \to A$ where $g \circ f = i_A$

We want to show that $g = f^{-1}$.

So we must show that

$$g(c) = f^{-1}(c) \text{ for all } c \in C.$$

Let $c \in C$.

Then, $f^{-1}(c) = a$ where

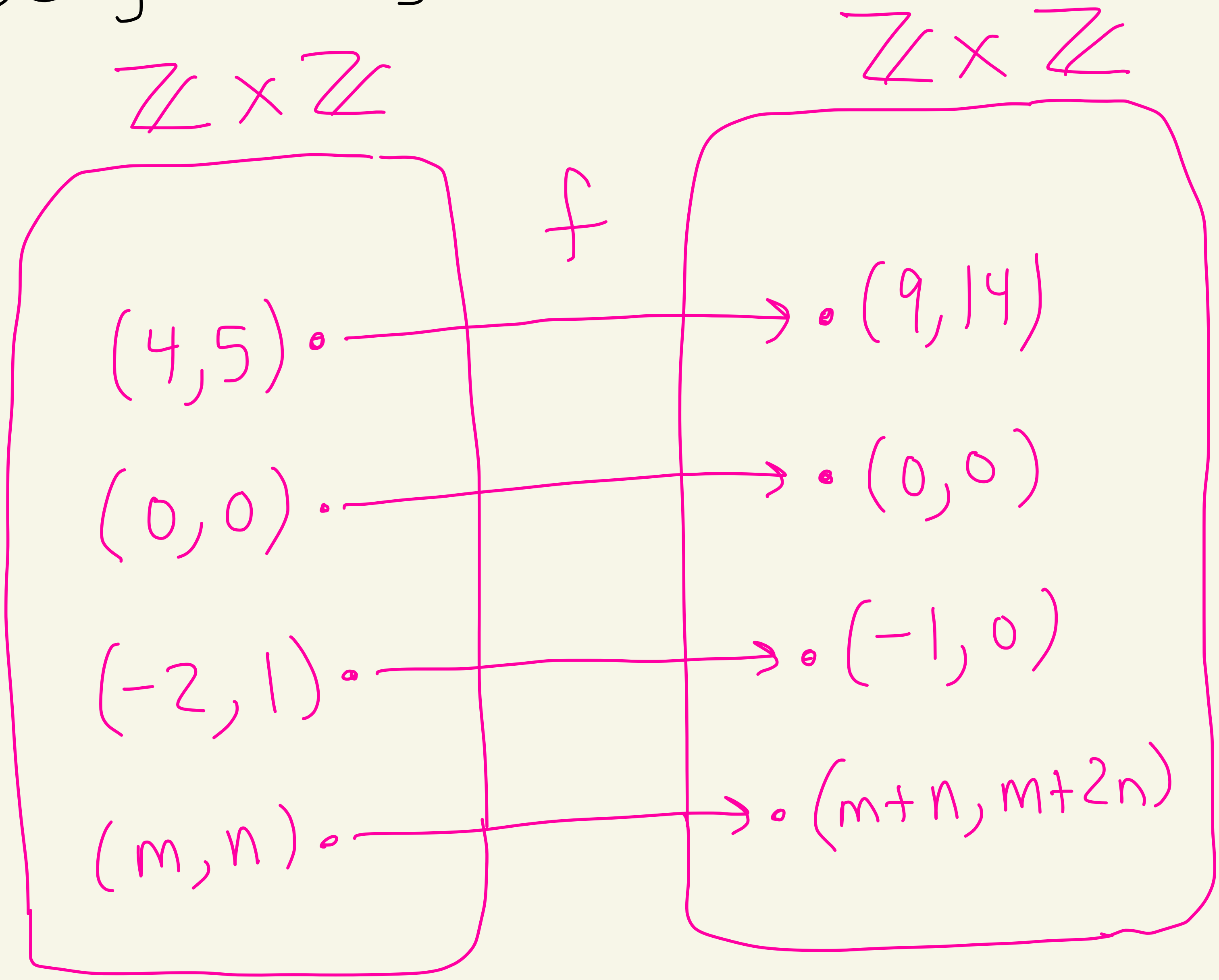
$$a \in A \text{ and } f(a) = c.$$

Then,

$$\begin{aligned} g(c) = g(f(a)) &= (g \circ f)(a) \\ &\overset{\text{assumption } g \circ f = i_A}{=} i_A(a) \\ &= a \\ &= f^{-1}(c) \end{aligned}$$

Thus, $g = f^{-1}$. $\blacksquare$

Ex: Let $f:\mathbb{Z}\times\mathbb{Z}\to\mathbb{Z}\times\mathbb{Z}$ be given by $f(m,n)=(m+n,m+2n)$



$f(4,5)=(4+5,\,4+2\cdot5)=(9,14)$

$f(-2,1)=(-2+1,\,-2+2\cdot1)=(-1,0)$

$\boxed{\text{Claim: } f \text{ is one-to-one}}$

proof:

Suppose $f(m_1, n_1) = f(m_2, n_2)$ where $(m_1, n_1), (m_2, n_2) \in \mathbb{Z} \times \mathbb{Z}$.

We need to show that $(m_1, n_1) = (m_2, n_2)$.

Since $f(m_1, n_1) = f(m_2, n_2)$ we know that $(m_1 + n_1, m_1 + 2n_1) = (m_2 + n_2, m_2 + 2n_2)$.

Thus,

$$\boxed{\begin{aligned} m_1 + n_1 &= m_2 + n_2 & (1) \\ m_1 + 2n_1 &= m_2 + 2n_2 & (2) \end{aligned}}$$

Calculating ② − ① we get that $n_1 = n_2$.

Thus we get

$$m_1 + n_2 = m_1 + n_1 = m_2 + n_2$$

$n_2 = n_1$ eqn ①

Subtract n_2 from both sides to get $m_1 = m_2$.

Thus, $(m_1, n_1) = (m_2, n_2)$.

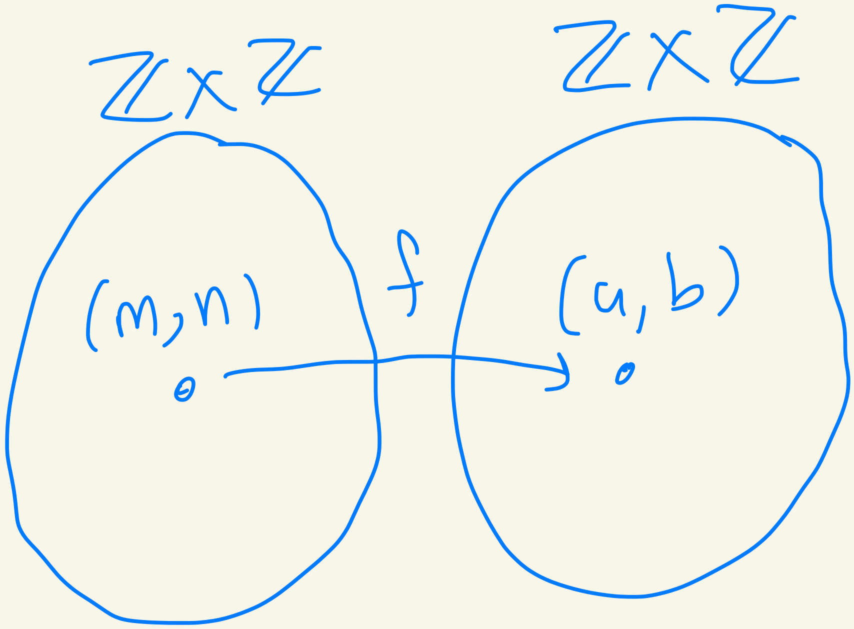
Thus, f is one-to-one.

claim 1

Claim 2: f is onto

Let $(a,b) \in \mathbb{Z} \times \mathbb{Z}$.

We must find $(m,n) \in \mathbb{Z} \times \mathbb{Z}$ where $f(m,n) = (a,b)$



That is, we need to solve

$$\underbrace{(m+n,\ m+2n)}_{f(m,n)} = (a,b).$$

So we need to solve

$$\boxed{\begin{aligned} m+n &= a \quad ① \\ m+2n &= b \quad ② \end{aligned}}$$

for m and n.

Calculating ② − ① you get that $n = b - a$.

Then,

$$\underset{\boxed{\text{eqn } ①}}{m =} a - n \underset{\boxed{n=b-a}}{=} a-(b-a) = 2a-b.$$

So, set $(m,n) = \underbrace{(2a-b,\ b-a)}_{\text{this is in } \mathbb{Z}\times\mathbb{Z} \text{ because } a,b\in\mathbb{Z}.}.$

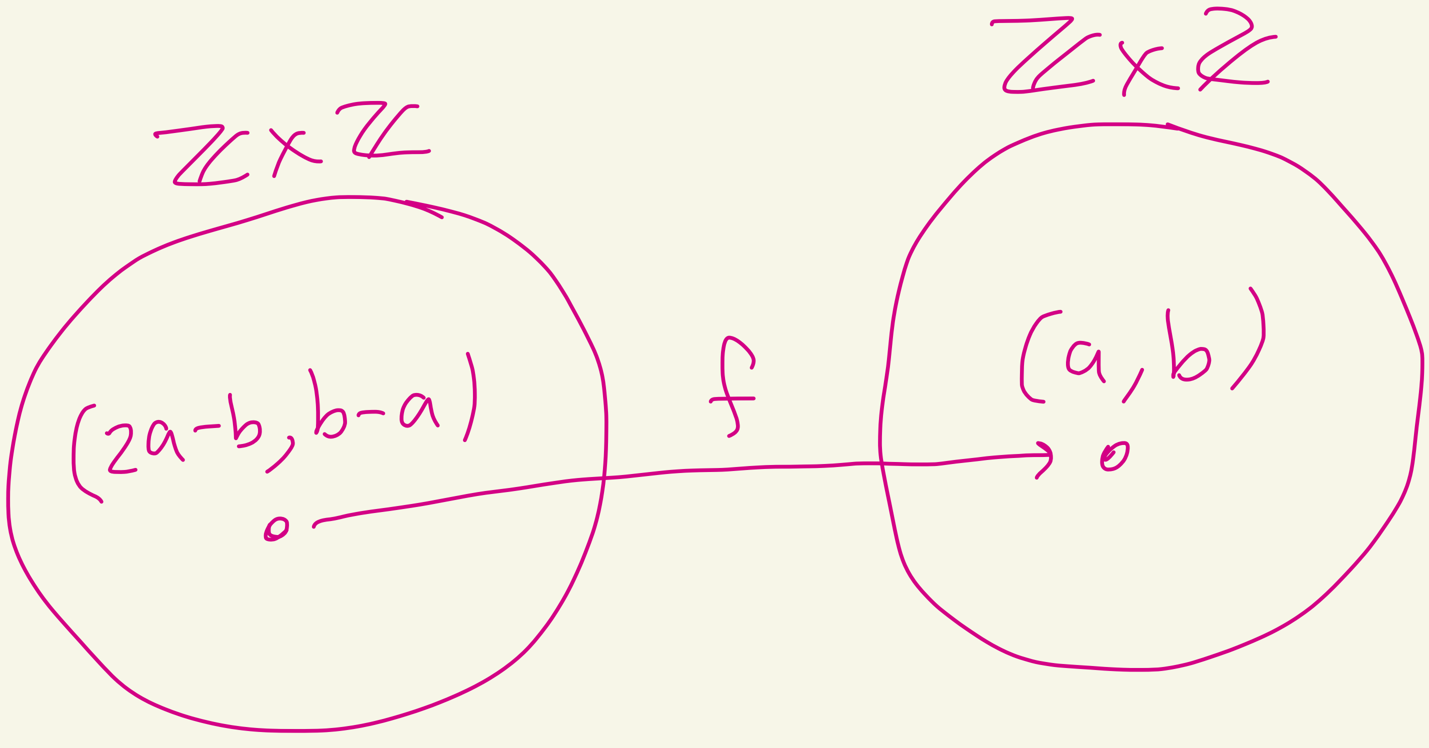
And we have that

$$f(m,n) = f(2a-b,\ b-a)$$

$$= (2a-b+b-a,\ 2a-b+2(b-a))$$

$$= (a, b)$$

Thus, f is onto.



Claim 2

From above we have that f is 1-1. Thus, f^{-1} exists.

And
$\text{domain}(f^{-1}) = \text{range}(f) = \mathbb{Z} \times \mathbb{Z}$

$\boxed{f \text{ is onto}}$

$\boxed{\text{Claim 3:}}$ Let $g : \mathbb{Z} \times \mathbb{Z} \to \mathbb{Z} \times \mathbb{Z}$ be defined by
$g(a,b) = (2a-b, b-a)$.
Then, $g = f^{-1}$.

Let's use thm from last time:
$f : A \to B$, f is 1-1, $C = \text{range}(f)$

If $g: C \to A$ and $g \circ f = i_A$ then $g = f^{-1}$

$\boxed{\text{proof of claim 3:}}$ We have

$$(g \circ f)(m,n) = g(f(m,n))$$

$$= g(m+n,\ m+2n)$$

$$= (2(m+n) - (m+2n),\ (m+2n) - (m+n))$$

$$= (m,n)$$

$$= i_{\mathbb{Z}\times\mathbb{Z}}(m,n).$$

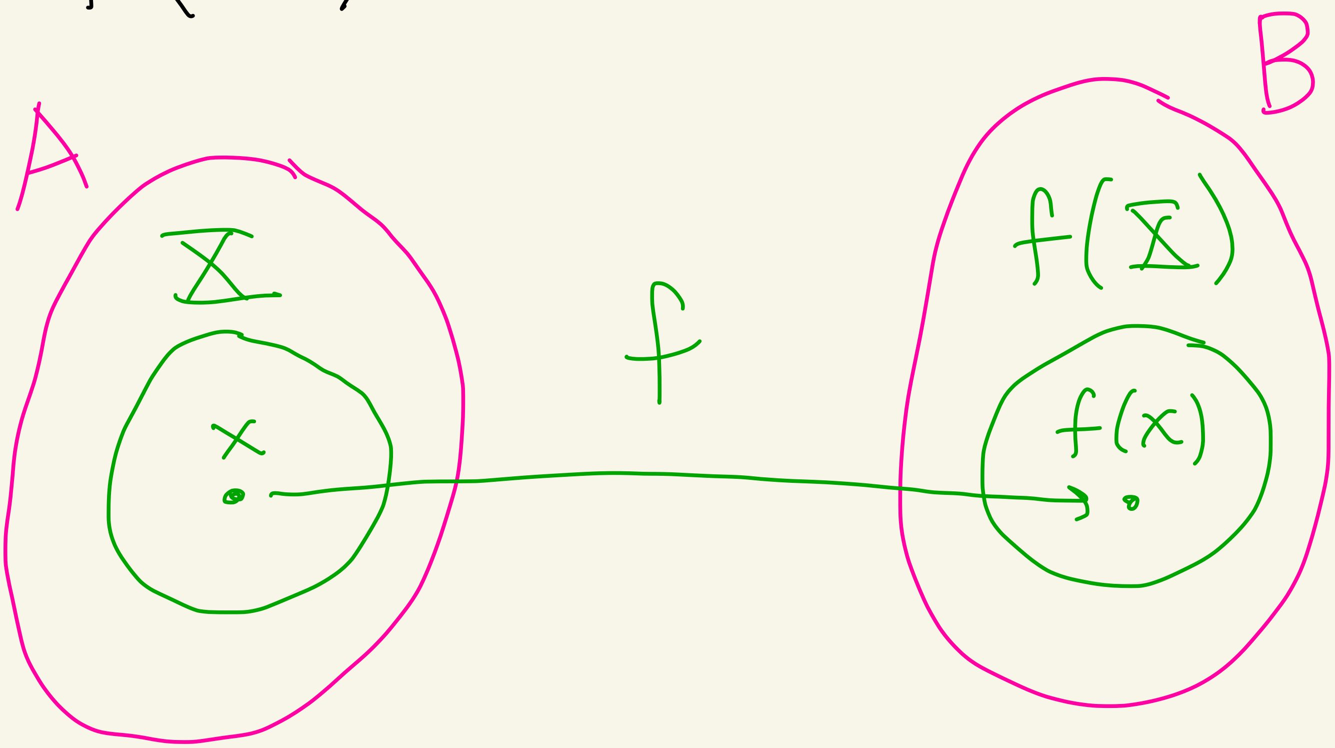
Since $g \circ f = i_{\mathbb{Z}\times\mathbb{Z}}$ we have $g = f^{-1}$!

$\boxed{\text{Claim 3}}$

Def: Let A and B be sets. Let $f: A \to B$.

① Let $X \subseteq A$.
The *image of X under f* is

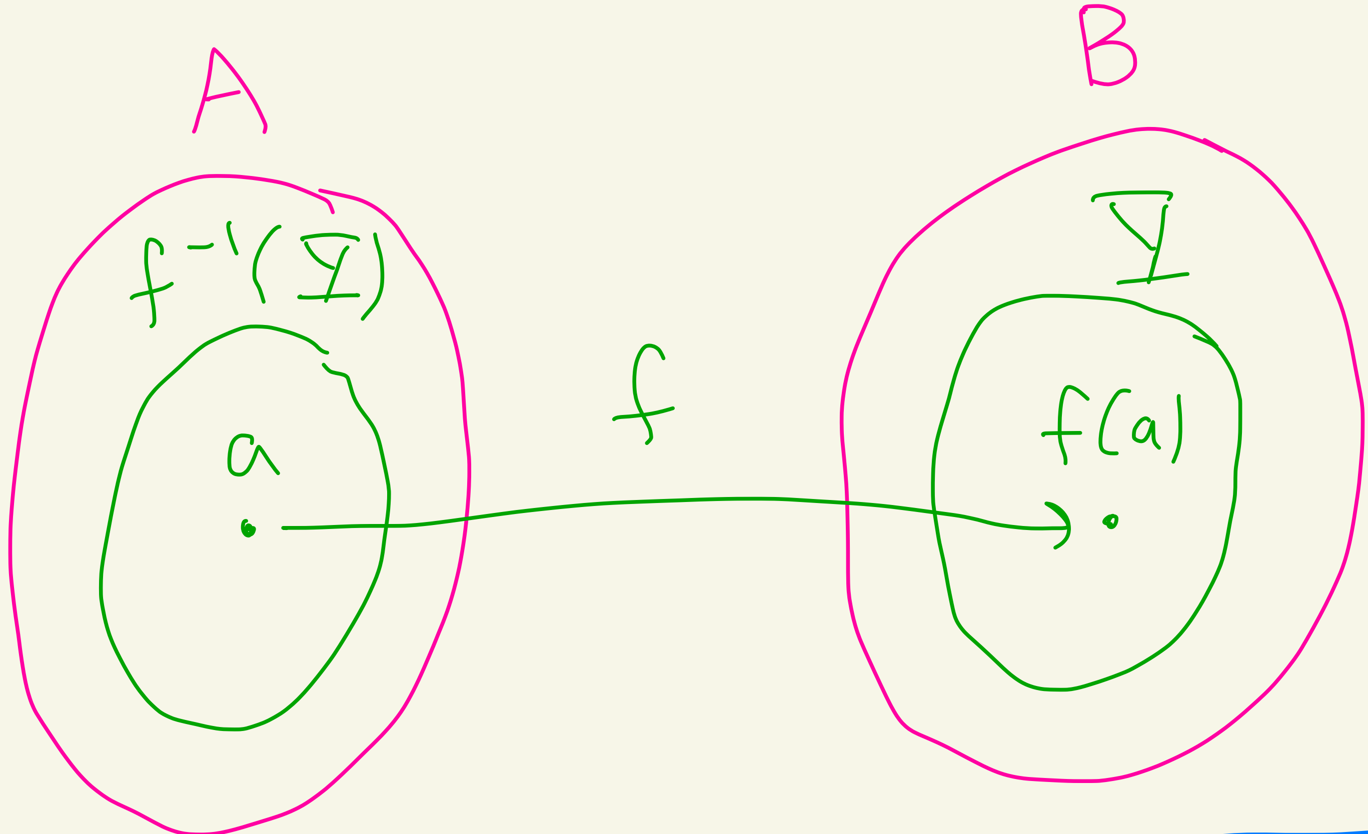
$$f(X) = \{ f(x) \mid x \in X \}$$



② Let $Y \subseteq B$.
The *inverse image of Y under f*

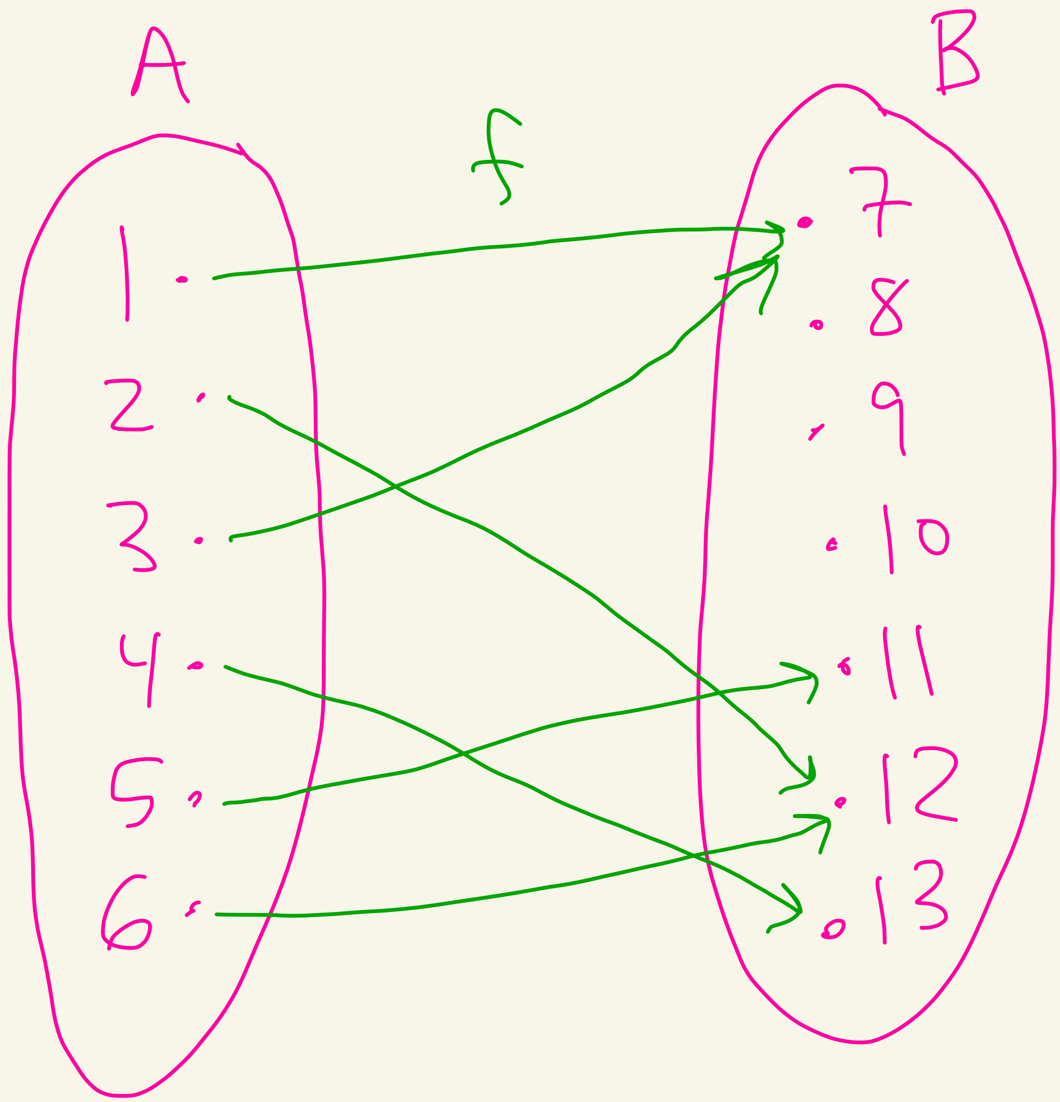
is the set

$$f^{-1}(Y) = \{ a \in A \mid f(a) \in Y \}$$



Note: We use f^{-1} notation, but it doesn't necessarily mean inverse function because f^{-1} might not exist

Ex: Consider the following function.



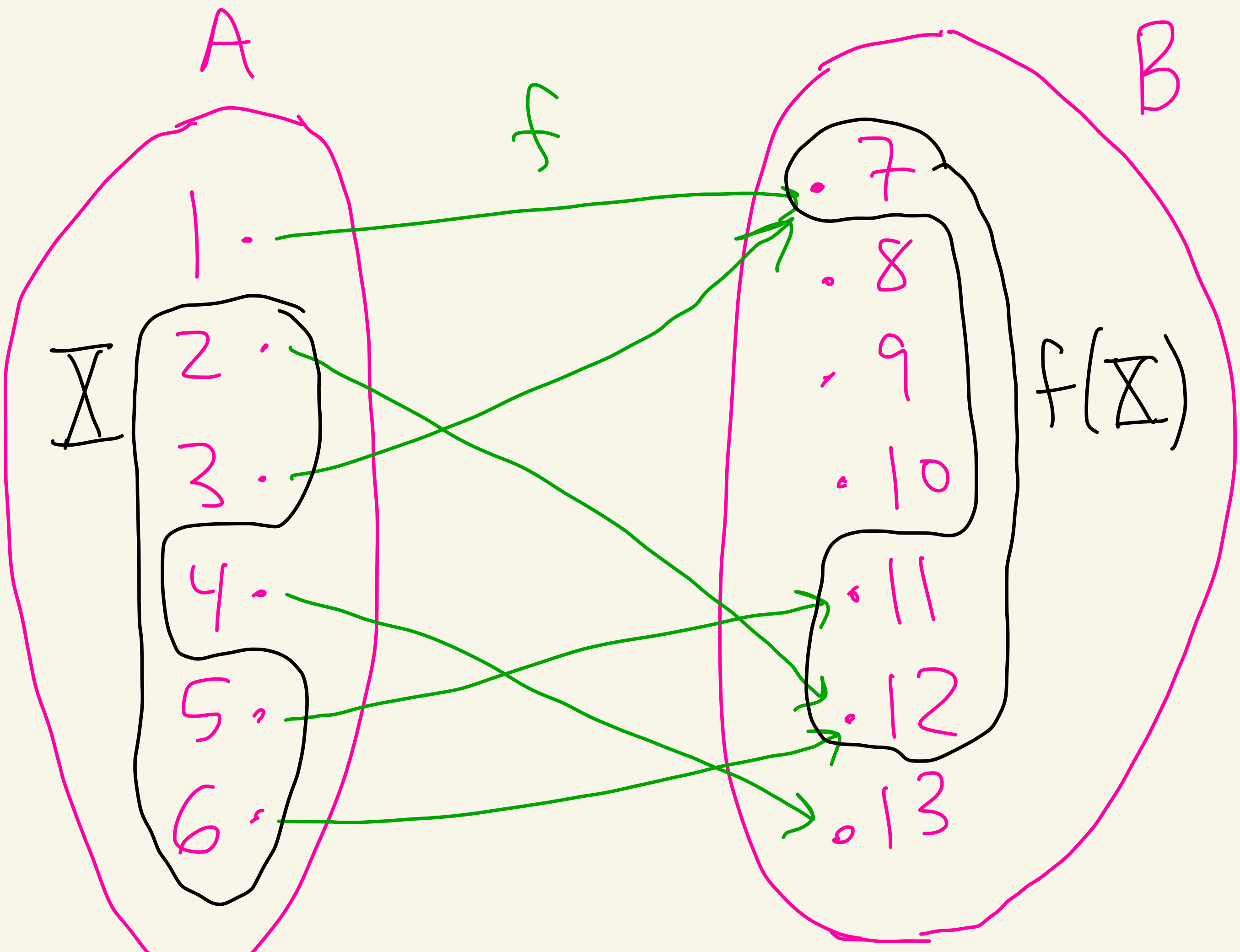
$f(1) = 7$
$f(2) = 12$
$f(3) = 7$
$f(4) = 13$
$f(5) = 11$
$f(6) = 12$

Let $X = \{2, 3, 5, 6\}$

Then,
$$f(X) = \{f(2), f(3), f(5), f(6)\}$$

$$= \{12, 7, 11, 12\}$$

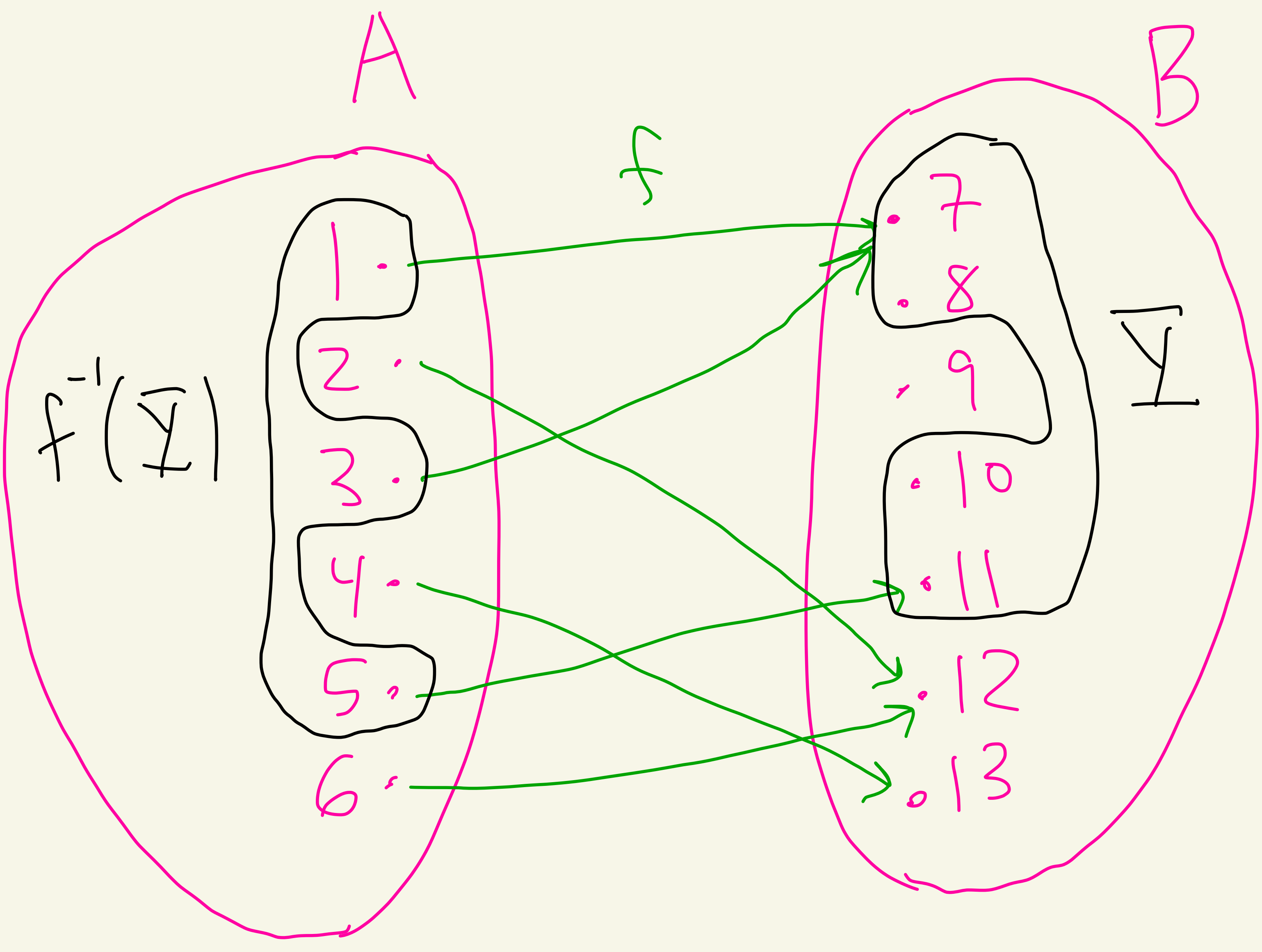
$$= \{7, 11, 12\}$$



Let $Y = \{7, 8, 10, 11\}$

Then,

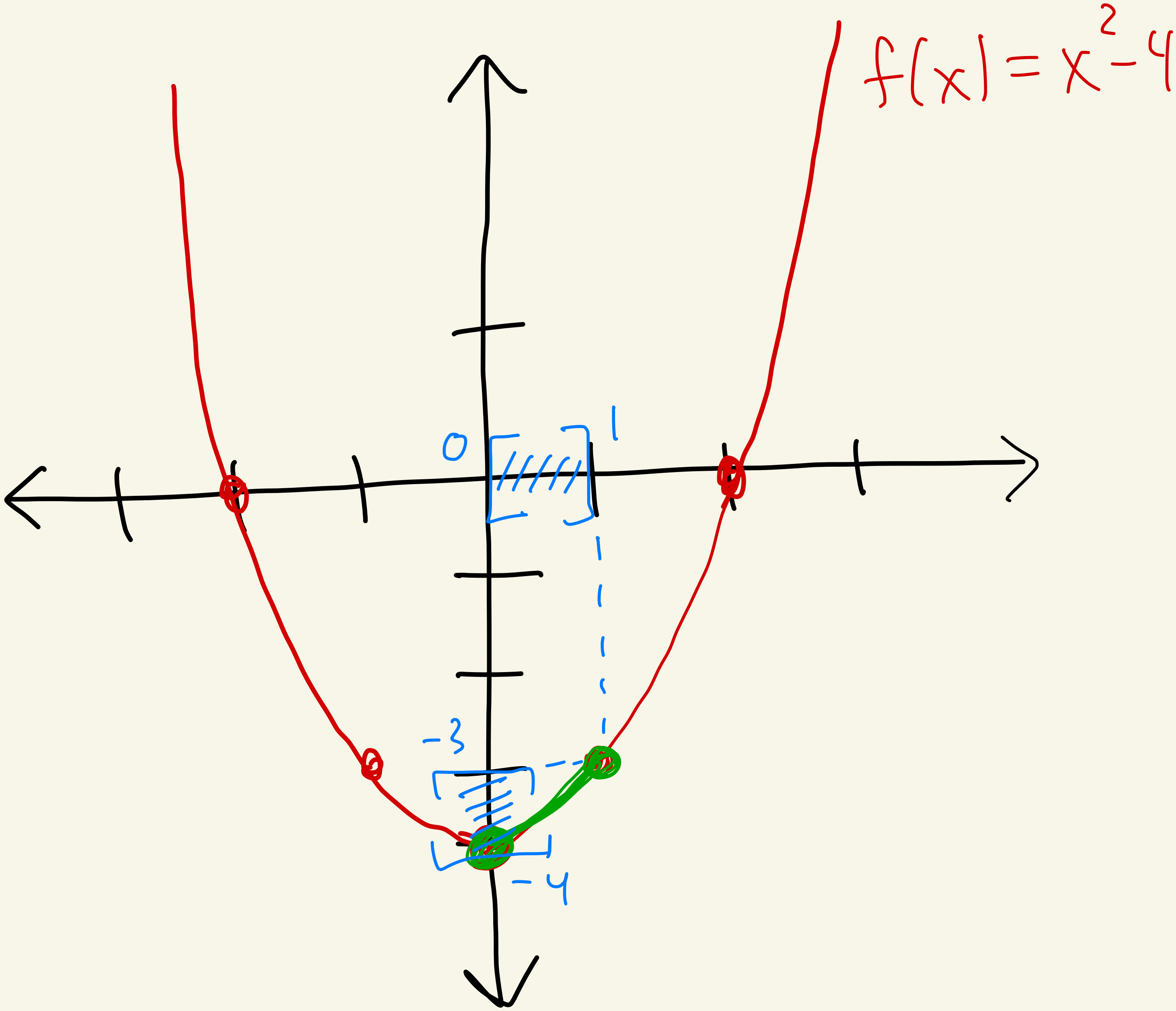
$$f^{-1}(Y) = \{1, 3, 5\}$$



$\boxed{\text{HW 4 problem modified}}$

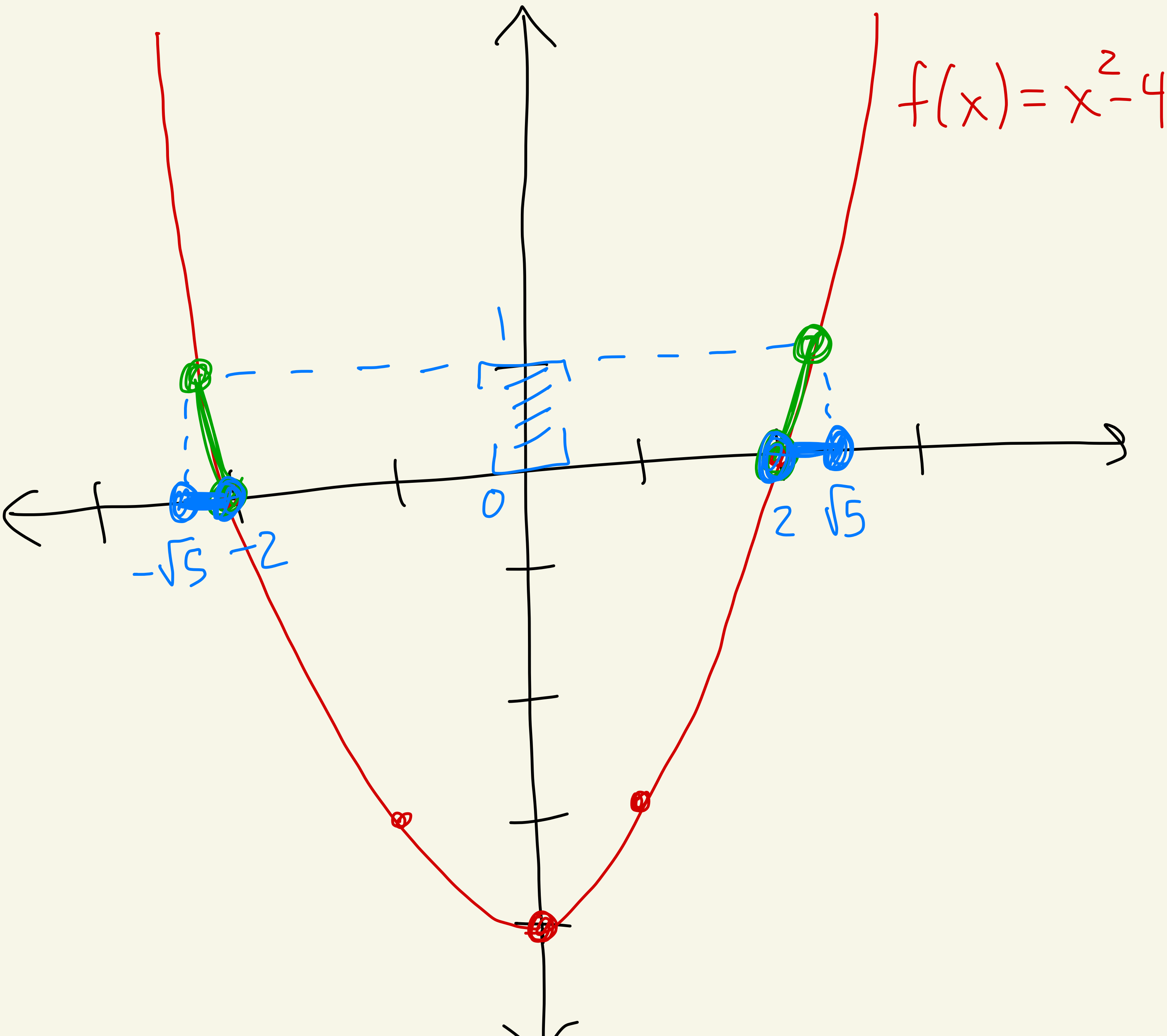
Let $f: \mathbb{R} \to \mathbb{R}$ where $f(x) = x^2 - 4$

(a) Calculate $f([0,1])$



$f([0,1]) = [-4,-3]$

(b) Calculate $f^{-1}([0,1])$

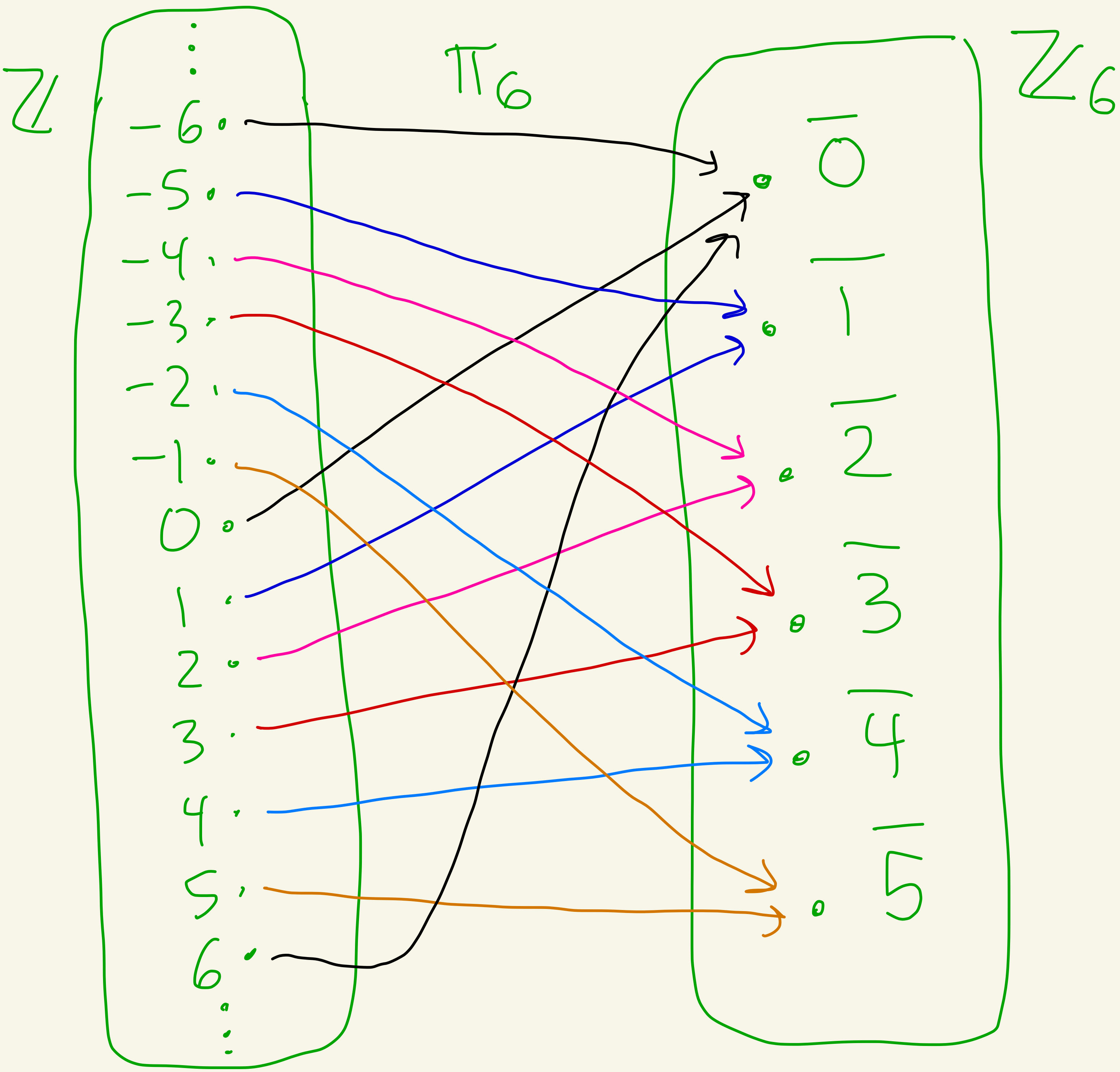


$$f^{-1}([0,1]) = [-\sqrt{5},-2] \cup [2,\sqrt{5}]$$

$\boxed{\text{HW 4 \#11}}$

$\pi_6 : \mathbb{Z} \to \mathbb{Z}_6$ where $\pi_6(x) = \overline{x}$

(a) Draw a picture of π_6



(b) Let $\mathbb{X} = \{1, 3, -5, 10, 102\}$

Then,

$$\pi_6(\mathbb{X}) = \{\pi_6(1), \pi_6(3), \pi_6(-5), \pi_6(10), \pi_6(102)\}$$

$$= \{\overline{1}, \overline{3}, \overline{-5}, \overline{10}, \overline{102}\}$$

$$= \{\overline{1}, \overline{3}, \overline{1}, \overline{4}, \overline{0}\}$$

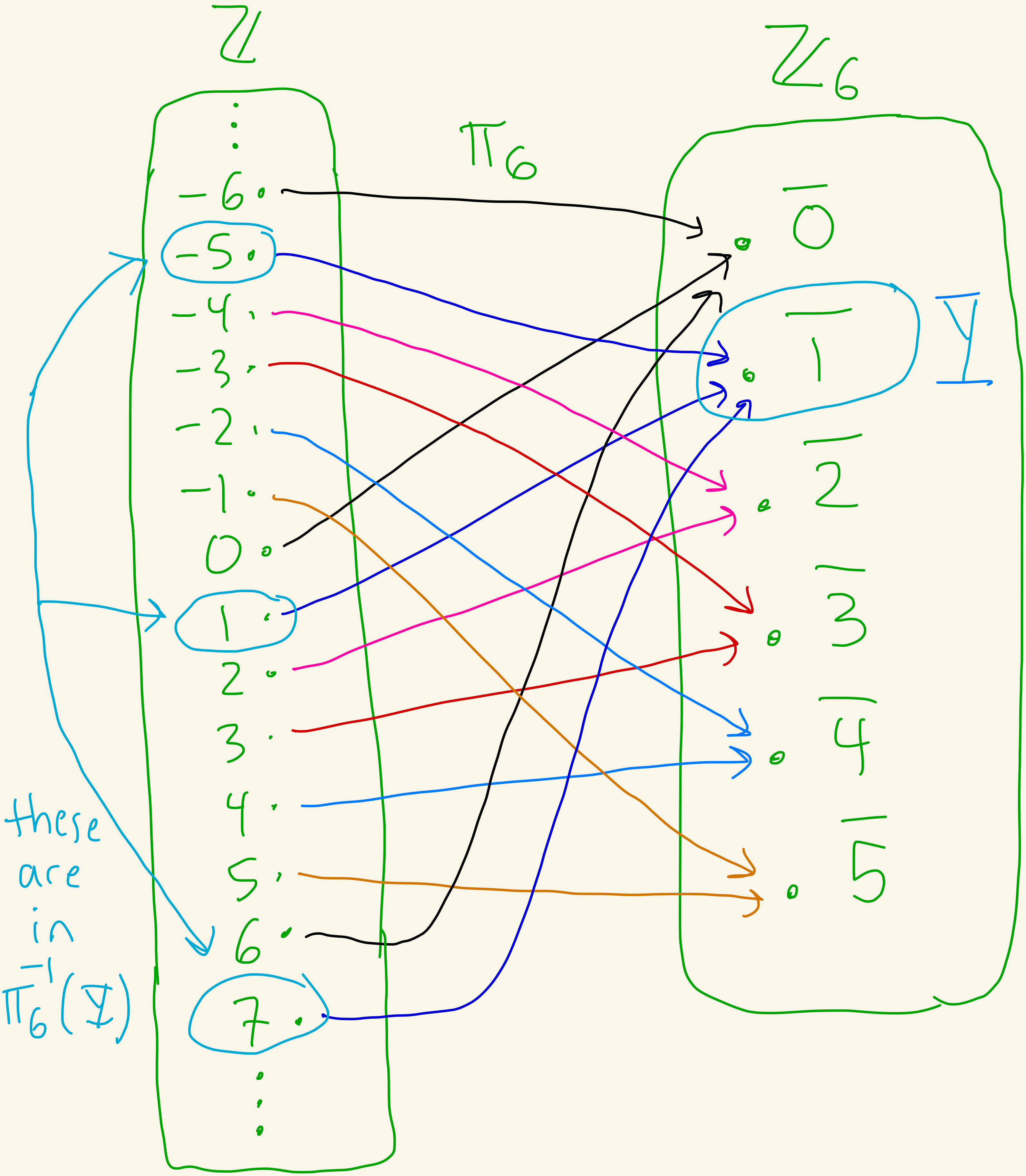
$$\begin{array}{r} 17 \\ 6\,\overline{)\,102} \\ -6 \\ \hline 42 \\ -42 \\ \hline 0 \end{array}$$

$$= \{\overline{0}, \overline{1}, \overline{3}, \overline{4}\}$$

(c) Let $\mathbb{Y} = \{\overline{1}\}$.

Let's calculate $\pi_6^{-1}(\mathbb{Y})$.

Let's take a look at the picture.



$\mathbb{Z}$
$\mathbb{Z}_6$
π_6
-6
-5
-4
-3
-2
-1
0
1
2
3
4
5
6
7
$\bar{0}$
$\bar{1}$
$\bar{2}$
$\bar{3}$
$\bar{4}$
$\bar{5}$
Y
these are in $\pi_6^{-1}(Y)$

Note: $-5, 1, 7 \in \pi_6^{-1}(\overline{1})$

And,

$$-5 = 6(-1) + 1$$
$$1 = 6(0) + 1$$
$$7 = 6(1) + 1$$

Also, $13 \in \pi_6^{-1}(\overline{1})$ and

$$13 = 6(2) + 1.$$

Claim: $\pi_6^{-1}(\overline{1}) = \{6k+1 \mid k \in \mathbb{Z}\}$

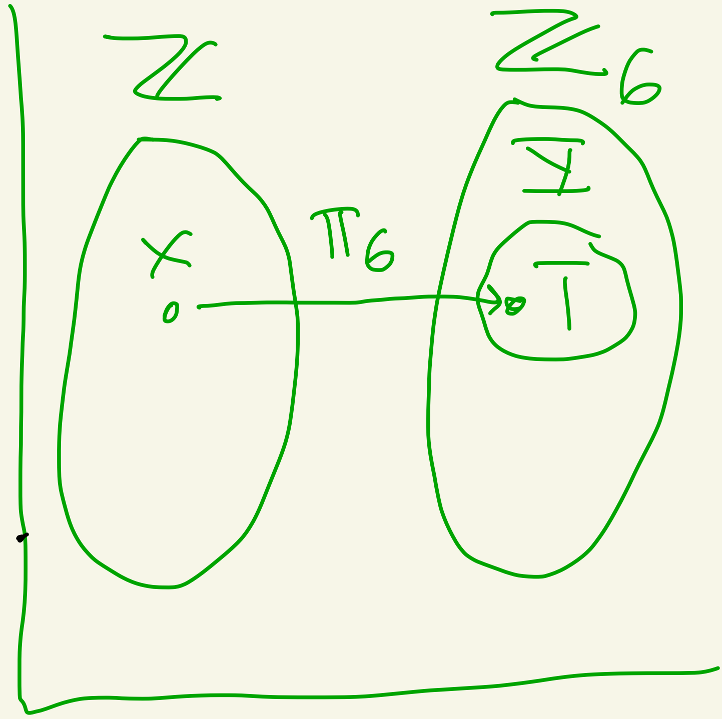
proof:

$(\subseteq)$: Let $x \in \pi_6^{-1}(\overline{1})$.

So, $\pi_6(x) \in Y$

Thus, $\pi_6(x) = \overline{1}$.

So, $\overline{x} = \overline{1}$ in $\mathbb{Z}_6$.



Then, $x \equiv 1 \pmod{6}$.

Thus, $6 \mid (x-1)$.

Hence, $x-1 = 6\ell$ where $\ell \in \mathbb{Z}$.

Therefore, $x = 6\ell + 1$.

Thus, $x \in \{6k+1 \mid k \in \mathbb{Z}\}$

Hence, $\pi_6^{-1}(Y) \subseteq \{6k+1 \mid k \in \mathbb{Z}\}$

$(\supseteq)$: Let $y \in \{6k+1 \mid k \in \mathbb{Z}\}$

So, $y = 6\ell + 1$ where $\ell \in \mathbb{Z}$.

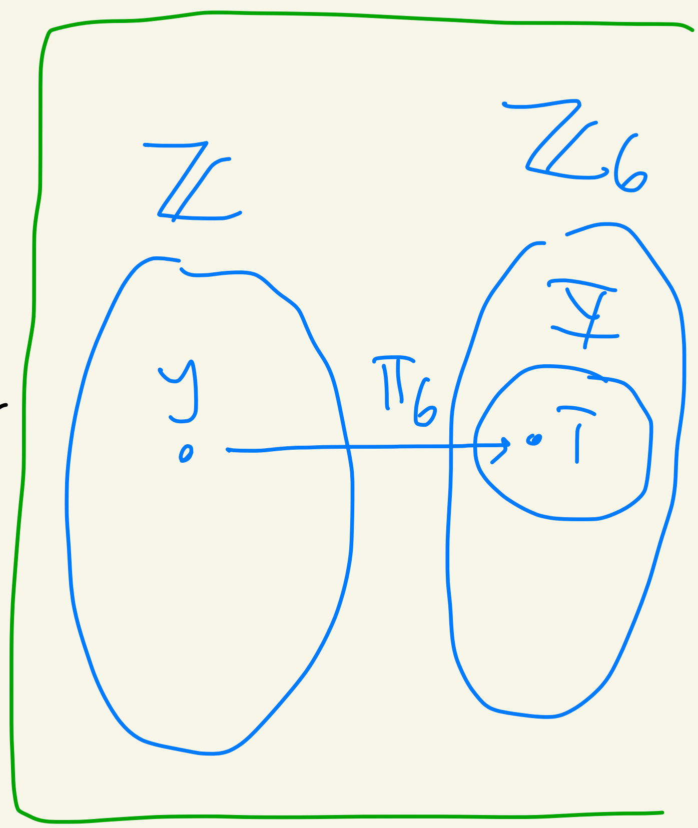
Then,

$$\pi_6(y) = \overline{y} = \overline{6\ell+1}$$
$$= \overline{6}\,\overline{\ell} + \overline{1}$$
$$= \overline{0}\,\ell + \overline{1}$$
$$= \overline{1}$$

$\overline{6} = \overline{0}$ in $\mathbb{Z}_6$

So, $\pi_6(y) \in \overline{Y}$.

Thus, $y \in \pi_6^{-1}(\overline{Y})$.

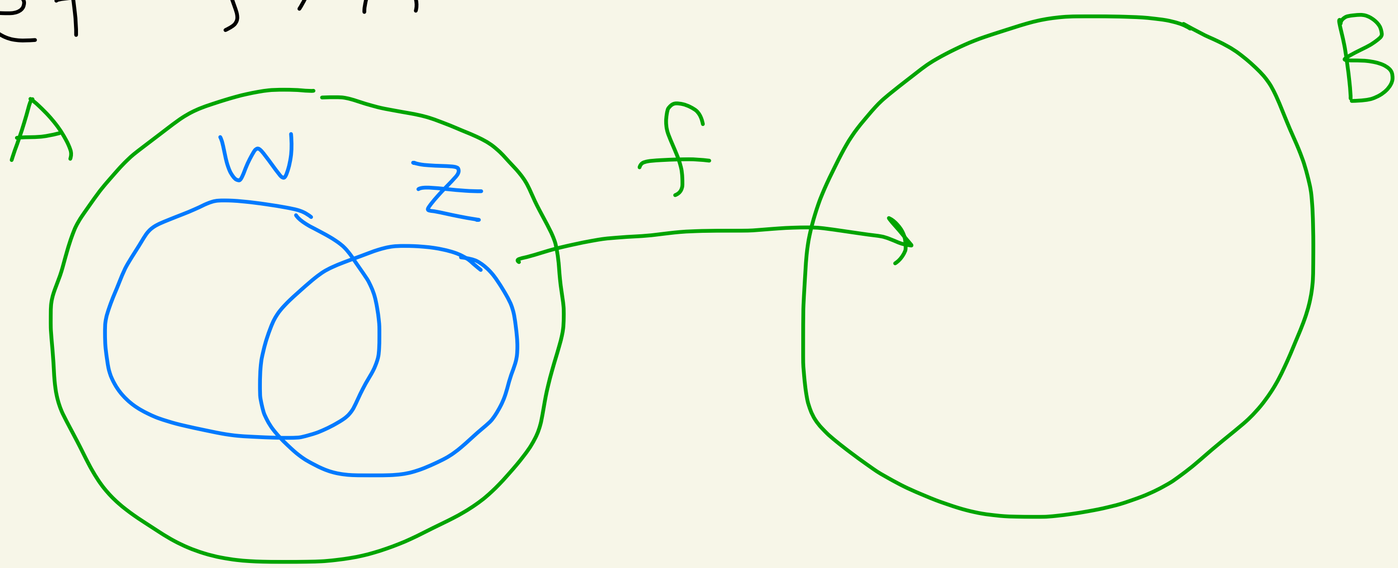


Therefore,

$\{6k+1 \mid k \in \mathbb{Z}\} \subseteq \pi_6^{-1}(\overline{Y})$.

By ($\subseteq$) and ($\supseteq$), $\pi_6^{-1}(\overline{Y}) = \{6k+1 \mid k \in \mathbb{Z}\}$ $\blacksquare$

Theorem: Let A, B, W, Z be sets where $W \subseteq A$ and $Z \subseteq A$. Let $f : A \to B$.



Then:

① $f(W \cup Z) = f(W) \cup f(Z)$ } HW #4

② $f(W \cap Z) \subseteq f(W) \cap f(Z)$

③ Give an example to show that $f(W \cap Z) = f(W) \cap f(Z)$ is not always true

} Hammack §2.6 #7, 8

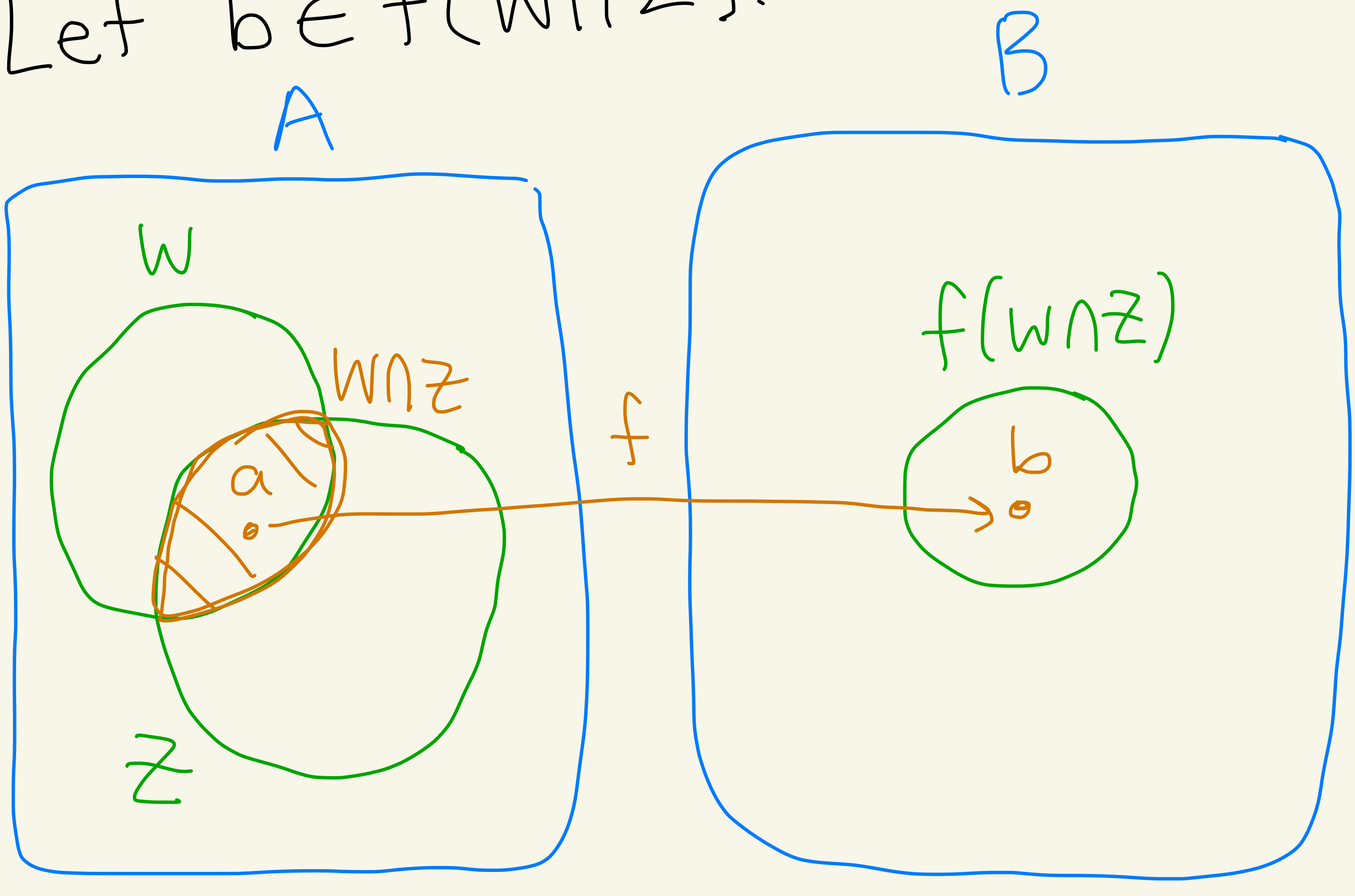
④ If $W \subseteq Z$, then $f(W) \subseteq f(Z)$

proof: Let's prove ②, ③, then ①, then ④

② We want to show that

$f(W \cap Z) \subseteq f(W) \cap f(Z)$.

Let $b \in f(W \cap Z)$.



Then there exists $a \in W \cap Z$ where $f(a) = b$.

Since $a \in W \cap Z$ we know
$a \in W$ and $a \in Z$.

Since $a \in W$ and $f(a) = b$
we know $b \in f(W)$

Since $a \in Z$ and $f(a) = b$
we know $b \in f(Z)$.

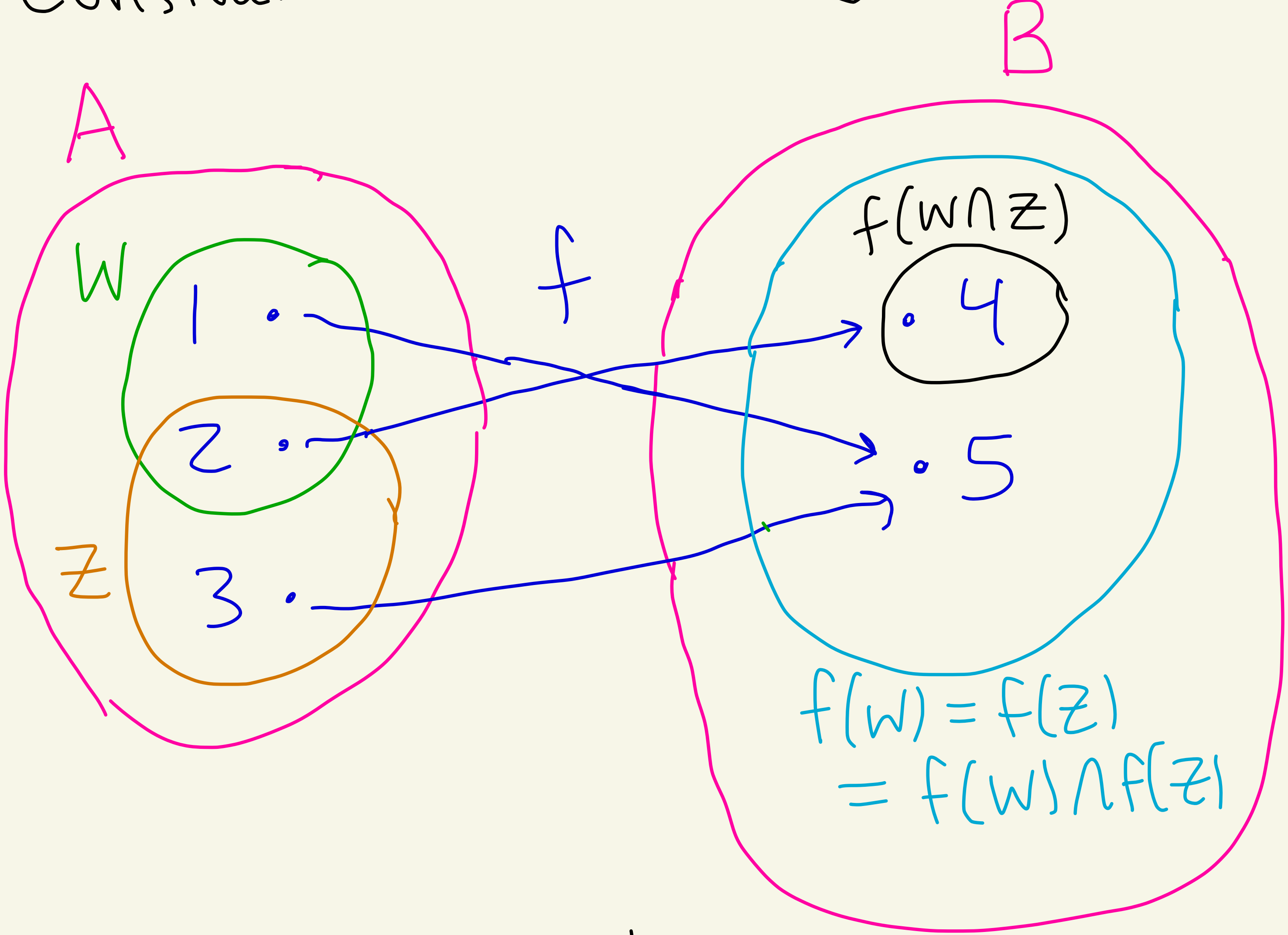
Thus, $b \in f(W) \cap f(Z)$.

Hence, $f(W \cap Z) \subseteq f(W) \cap f(Z)$.

---

3 Let's give an example
to show that
$f(W \cap Z) = f(W) \cap f(Z)$
is not always true.

Consider the following:



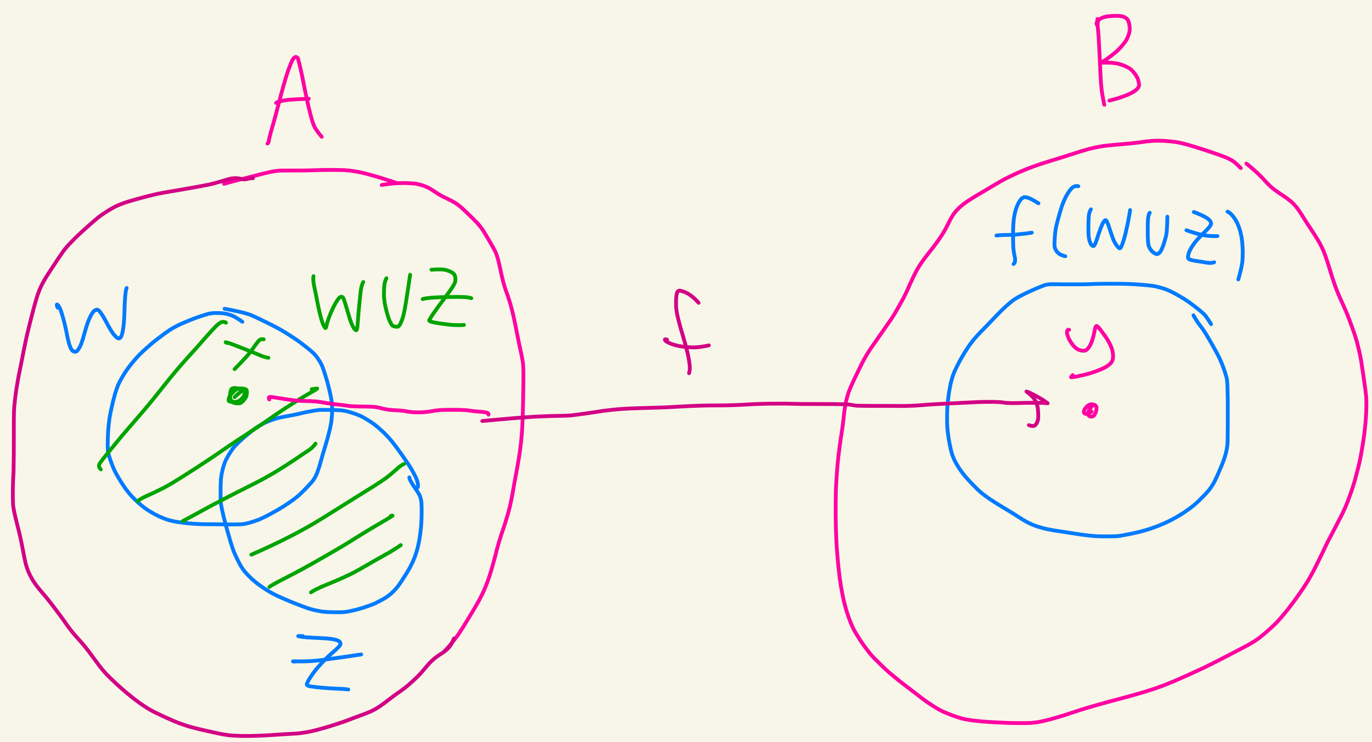
In this example,

$$f(W \cap Z) = \{4\} \neq \{4, 5\} = f(W) \cap f(Z)$$

① We want to show that

$$f(W \cup Z) = f(W) \cup f(Z)$$

$(\subseteq)$: Let $y \in f(W \cup Z)$.

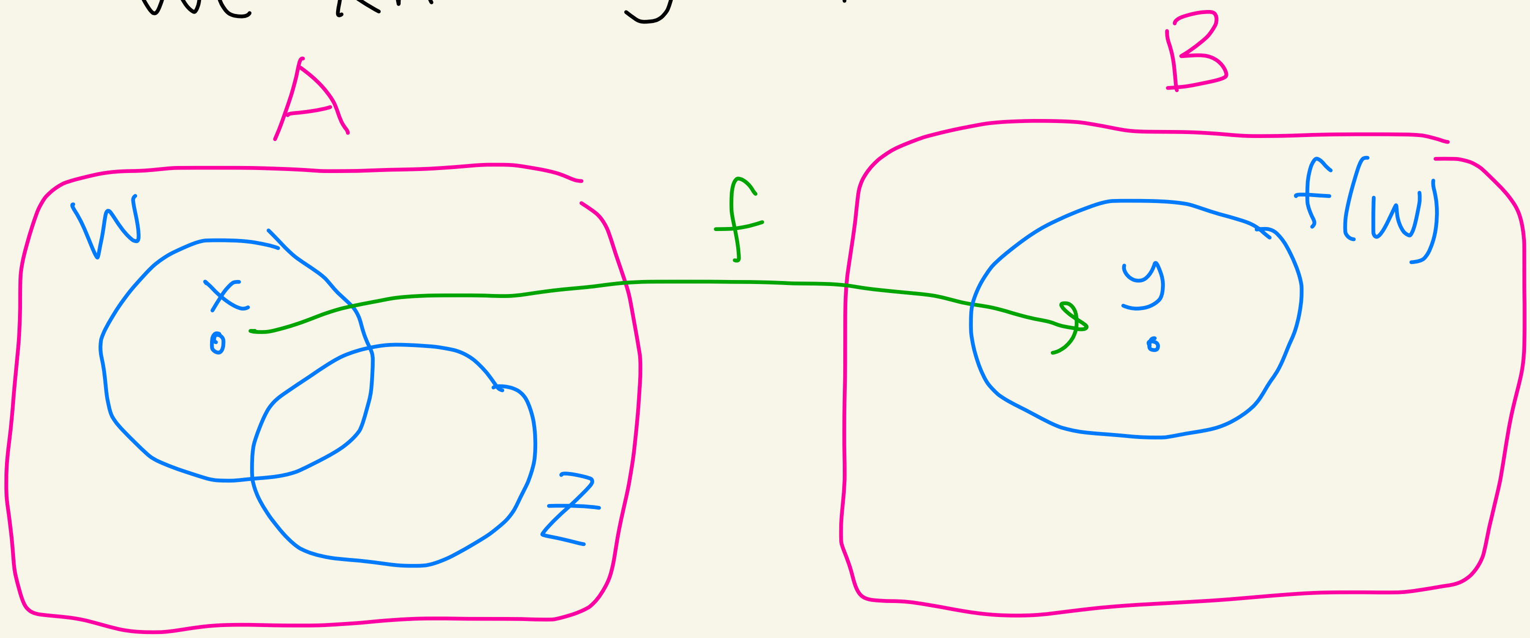


Then there exists $x \in W \cup Z$ where $f(x) = y$.

Since $x \in W \cup Z$ we know $x \in W$ or $x \in Z$.

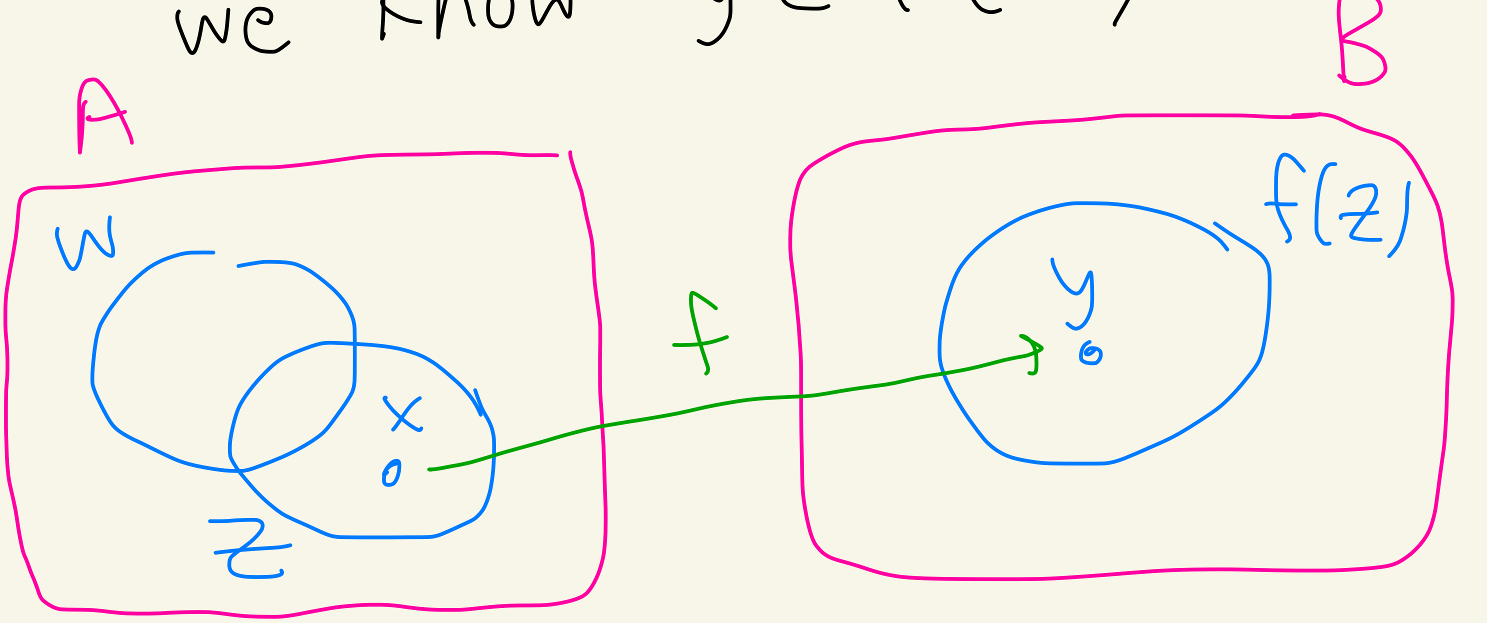
<u>case 1</u>: Suppose $x \in W$.

Then since $x \in W$ and $f(x) = y$ we know $y \in f(W)$



<u>Case 2</u>: Suppose $x \in Z$.

Then since $x \in Z$ and $f(x) = y$ we know $y \in f(Z)$.



So either $y \in f(W)$ or $y \in f(Z)$ from the two cases above.

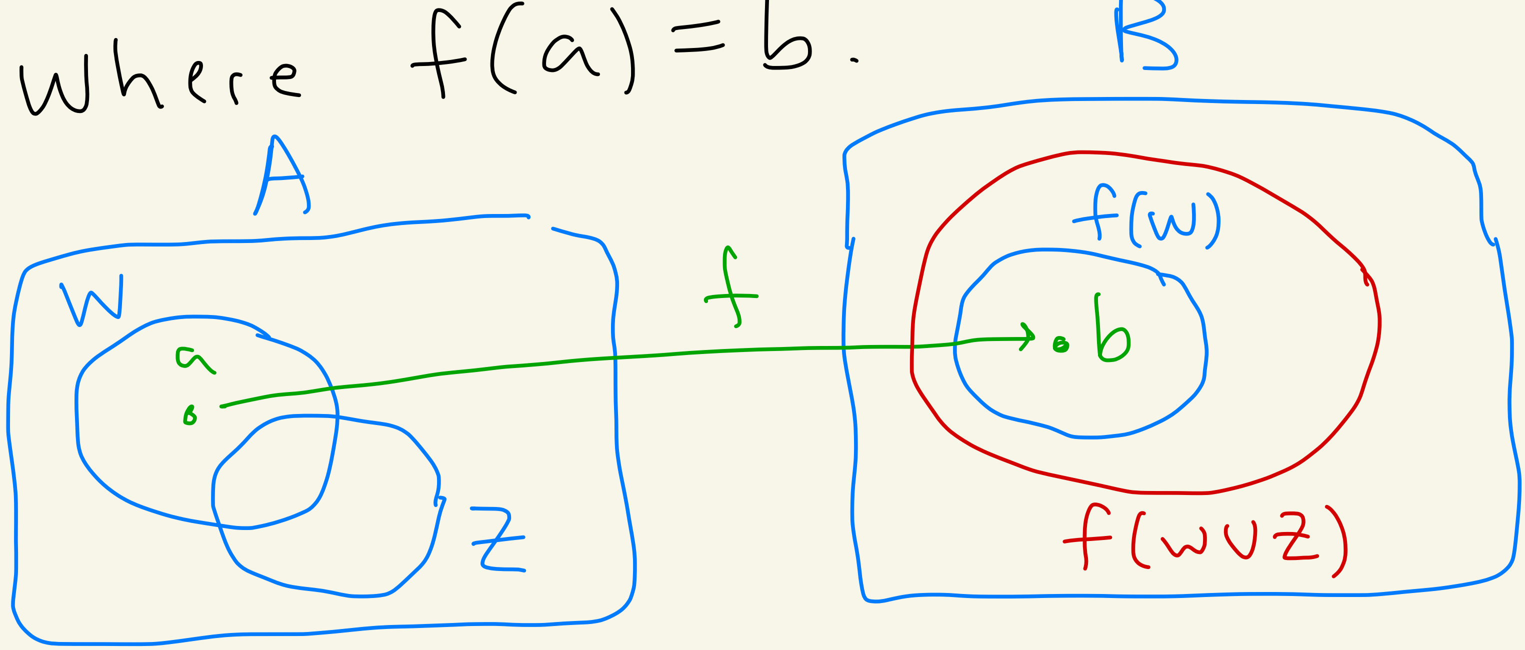
Thus, $y \in f(W) \cup f(Z)$.

($\supseteq$): Let $b \in f(W) \cup f(Z)$.

Then, $b \in f(W)$ or $b \in f(Z)$.

<u>Case 1</u>: Suppose $b \in f(W)$.

Then there exists $a \in W$ where $f(a) = b$.



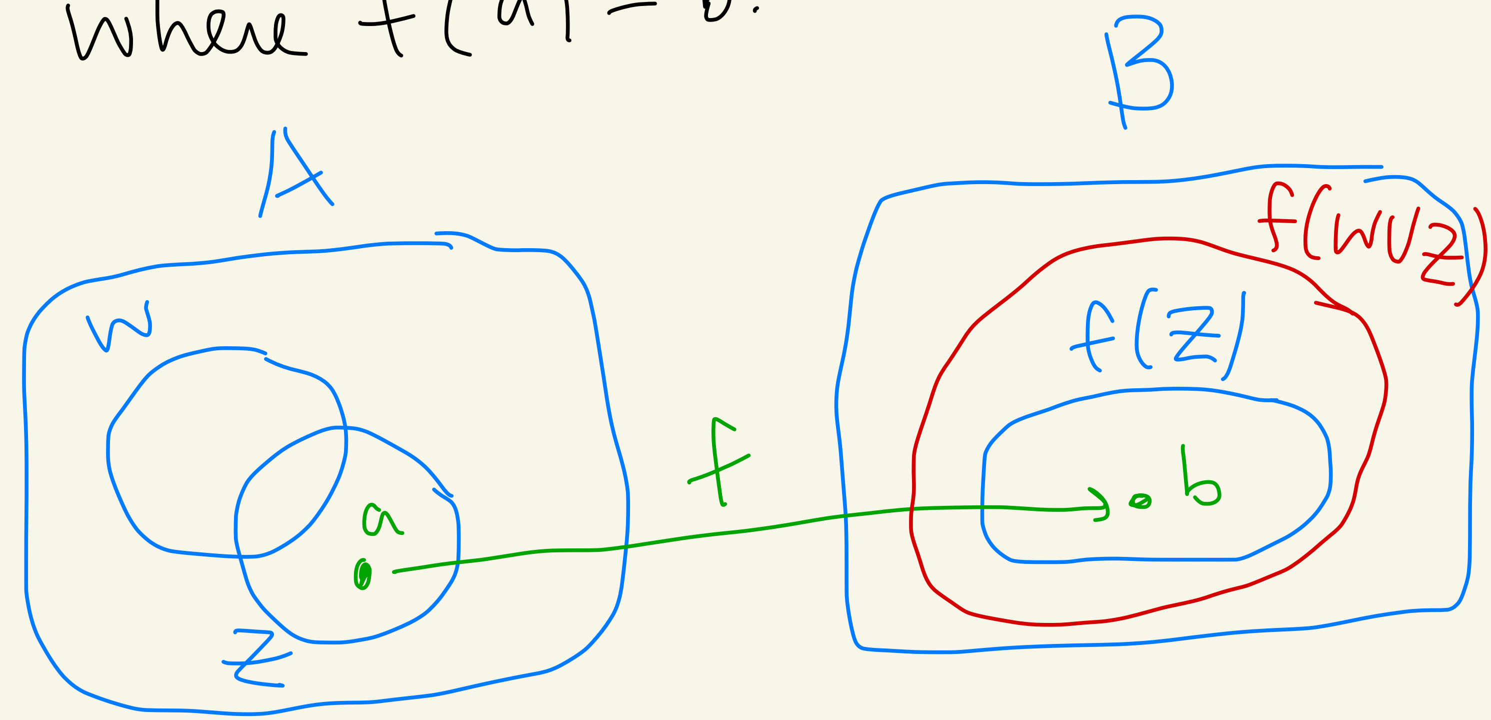
But $a \in W \subseteq W \cup Z$.

So, $a \in W \cup Z$ and $f(a) = b$.

Thus, $b \in f(W \cup Z)$.

<u>Case 2:</u> Suppose $b \in f(Z)$.

Then there exists $a \in Z$ where $f(a) = b$.



But $a \in Z \subseteq W \cup Z$.

So, $a \in W \cup Z$ and $f(a) = b$.

Thus, $b \in f(W \cup Z)$.

Therefore, in either case 1 or case 2 we get $b \in f(W \cup Z)$.

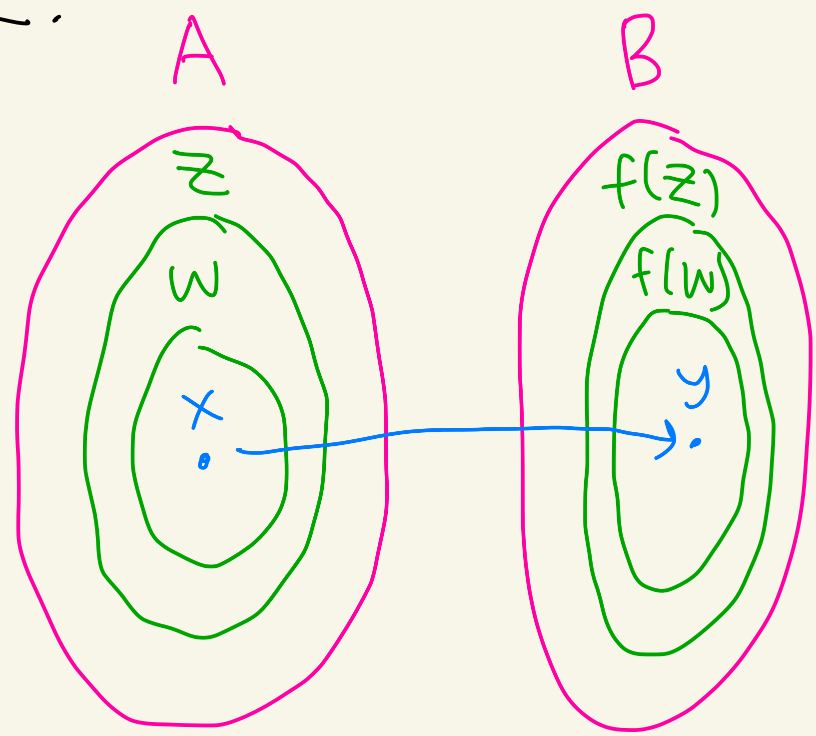
Thus, $f(W) \cup f(Z) \subseteq f(W \cup Z)$.

By, $(\subseteq)$ and $(\supseteq)$ we get
$$f(W \cup Z) = f(W) \cup f(Z).$$

---

4 Suppose $W \subseteq Z$.

Let $y \in f(W)$.
Then, there exists $x \in W$ with $f(x) = y$.



Since $W \subseteq Z$ and $x \in W$

We know that $x \in Z$.

Since $x \in Z$ and $f(x) = y$ we know that $y \in f(Z)$.

We have shown that $f(W) \subseteq f(Z)$.

Recall:

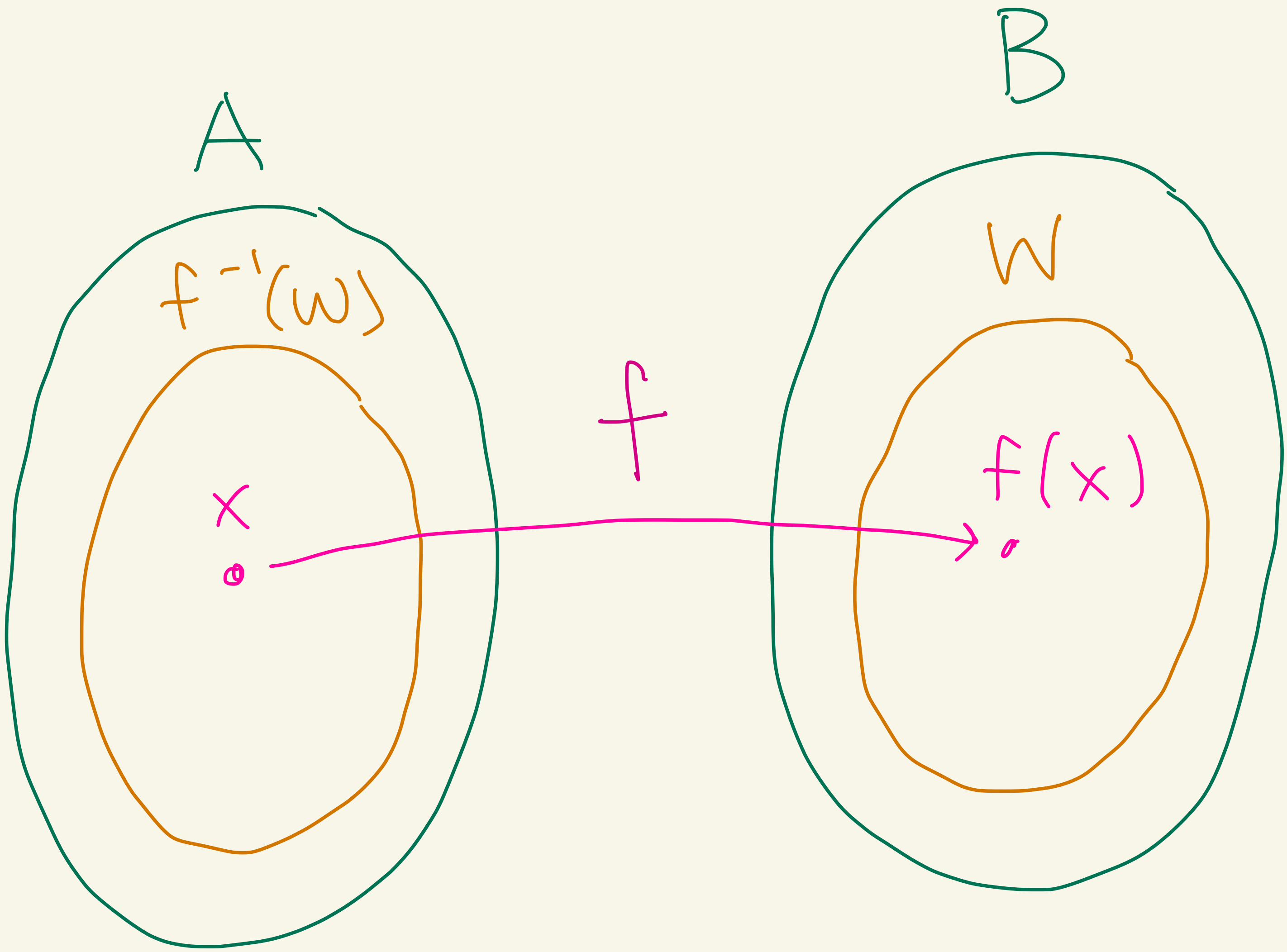
$f: A \to B$

$W \subseteq B$

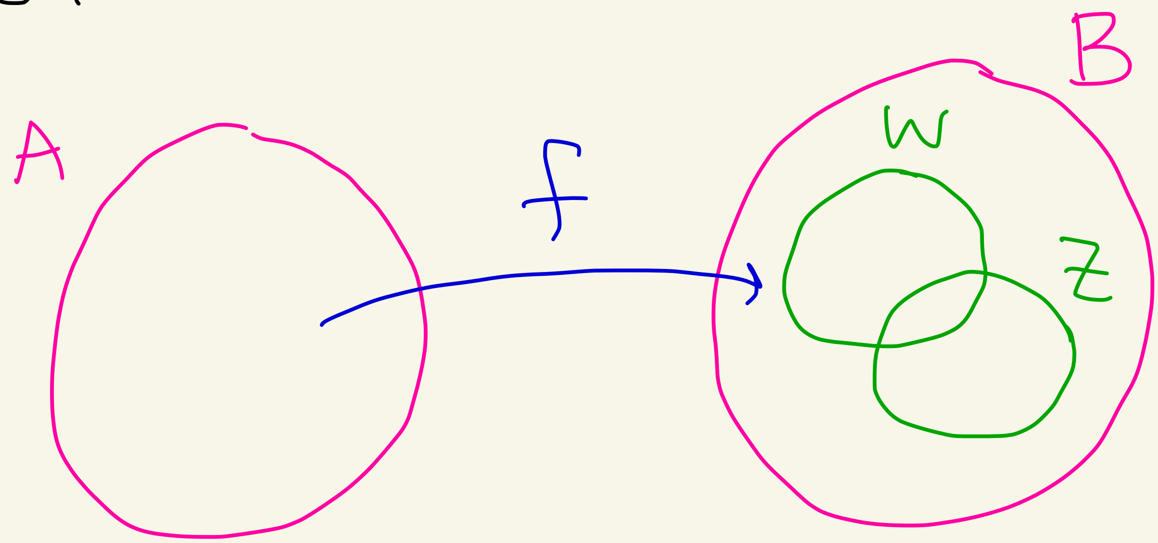
$$f^{-1}(W) = \{ x \in A \mid f(x) \in W \}$$

Key:

$x \in f^{-1}(W)$ means $f(x) \in W$



Theorem: Let A, B be sets.
Let $f: A \to B$.
Let $W \subseteq B$ and $Z \subseteq B$.



Then:

1. $f^{-1}(W \cap Z) = f^{-1}(W) \cap f^{-1}(Z)$
2. $f^{-1}(W \cup Z) = f^{-1}(W) \cup f^{-1}(Z)$
3. $A - f^{-1}(W) = f^{-1}(B - W)$
4. If $W \subseteq Z$, then $f^{-1}(W) \subseteq f^{-1}(Z)$

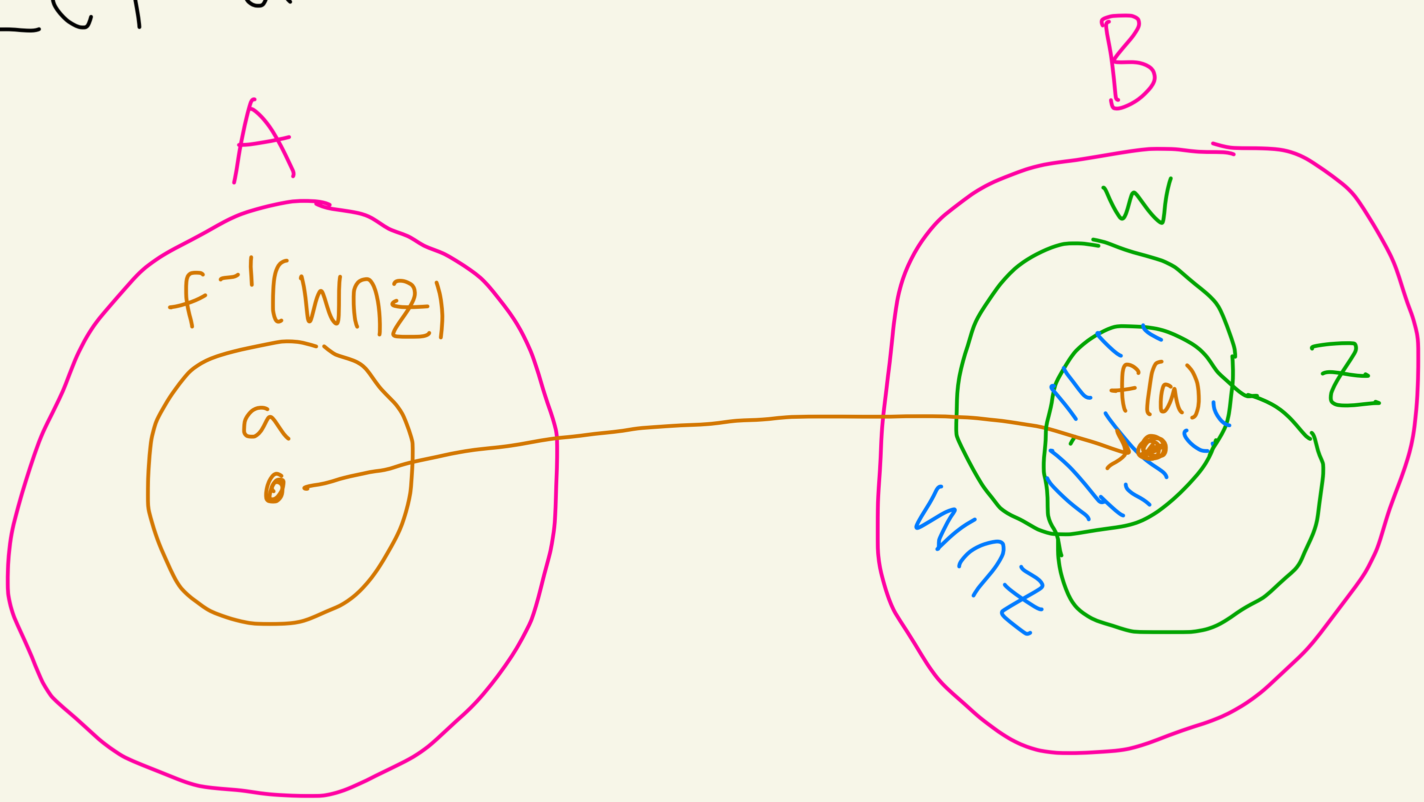
proof:

① Let's show that

$f^{-1}(W\cap Z)=f^{-1}(W)\cap f^{-1}(Z)$.

$\boxed{\subseteq}$:

Let $a\in f^{-1}(W\cap Z)$.



Then, $f(a)\in W\cap Z$.

So, $f(a)\in W$ and $f(a)\in Z$.

Thus, $a \in f^{-1}(W)$ and $a \in f^{-1}(Z)$.

Therefore, $a \in f^{-1}(W) \cap f^{-1}(Z)$.

$\boxed{\supseteq}$:

Let $x \in f^{-1}(W) \cap f^{-1}(Z)$.

Then, $x \in f^{-1}(W)$ and $x \in f^{-1}(Z)$.

So, $f(x) \in W$ and $f(x) \in Z$

Thus, $f(x) \in W \cap Z$.

So, $x \in f^{-1}(W \cap Z)$.

By $(\subseteq)$ and $(\supseteq)$ we get

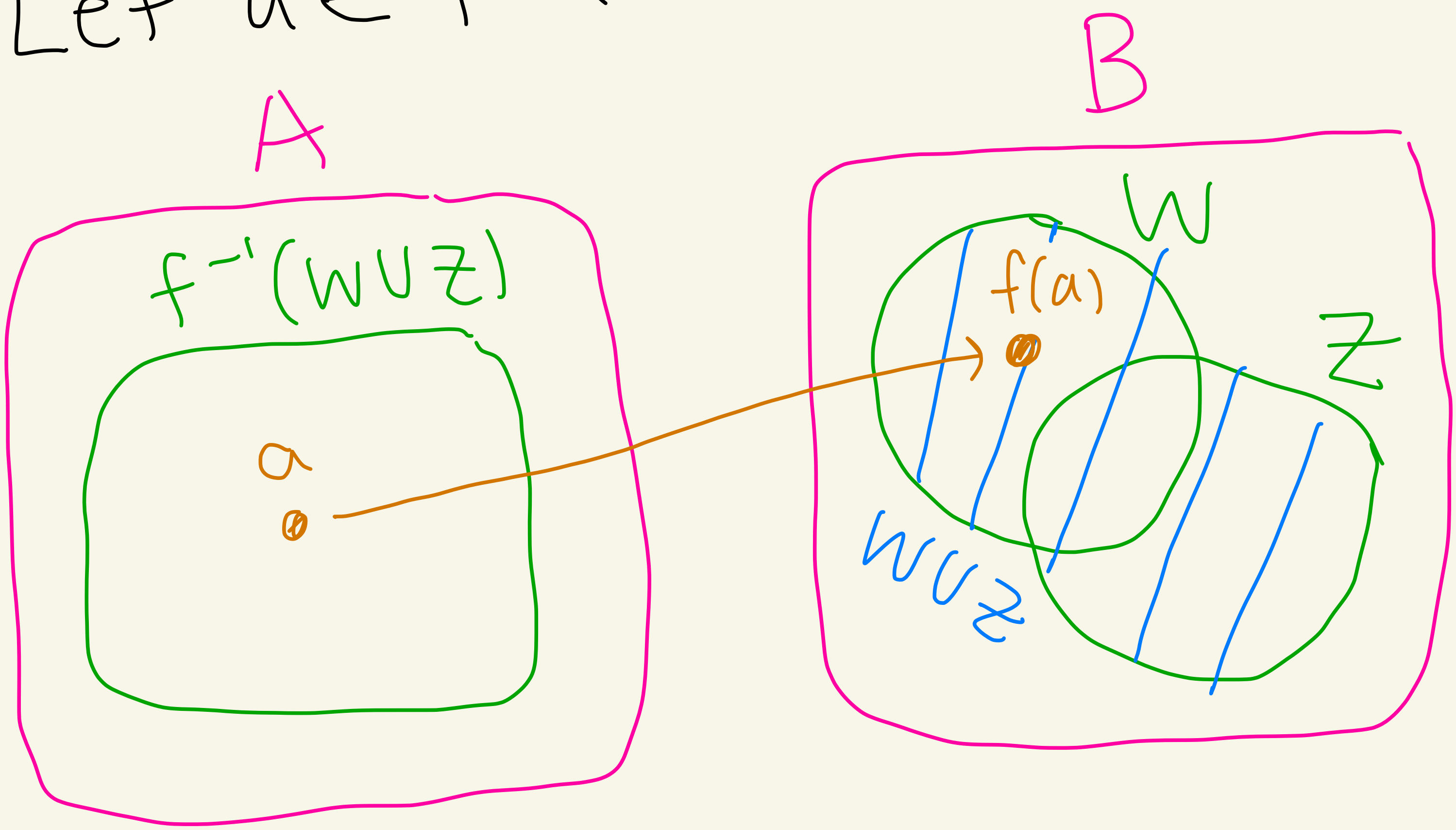
$$f^{-1}(W \cap Z) = f^{-1}(W) \cap f^{-1}(Z).$$

② Let's show that

$$f^{-1}(W \cup Z) = f^{-1}(W) \cup f^{-1}(Z)$$

$\boxed{\subseteq}$:

Let $a \in f^{-1}(W \cup Z)$.



So, $f(a) \in W \cup Z$.

Thus, $f(a) \in W$ or $f(a) \in Z$.

Hence, $a \in f^{-1}(W)$ or $a \in f^{-1}(Z)$.

Ergo, $a \in f^{-1}(W) \cup f^{-1}(Z)$.

Thus, $f^{-1}(W \cup Z) \subseteq f^{-1}(W) \cup f^{-1}(Z)$.

$\boxed{\supseteq}$:

Let $x \in f^{-1}(W) \cup f^{-1}(Z)$.

Then, $x \in f^{-1}(W)$ or $f^{-1}(Z)$.

So, $f(x) \in W$ or $f(x) \in Z$.

Hence, $f(x) \in W \cup Z$.

Thus, $x \in f^{-1}(W \cup Z)$.

Hence, $f^{-1}(W) \cup f^{-1}(Z) \subseteq f^{-1}(W \cup Z)$

By ($\subseteq$) and ($\supseteq$) we have that $f^{-1}(W \cup Z) = f^{-1}(W) \cup f^{-1}(Z)$.

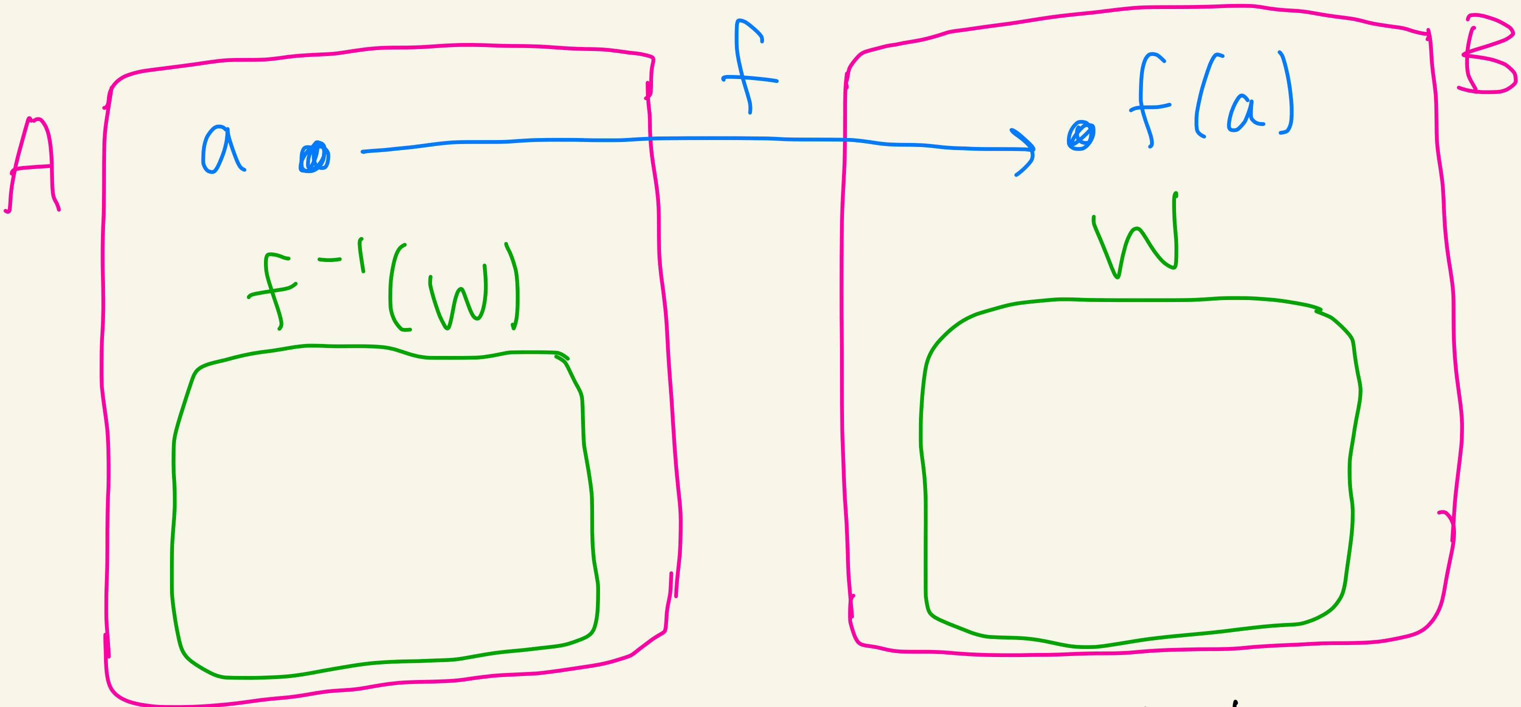
$\boxed{\begin{array}{l} \text{Iff version of } ②: \\ a \in f^{-1}(W \cup Z) \\ \text{iff } f(a) \in W \cup Z \\ \text{iff } f(a) \in W \text{ or } f(a) \in Z \\ \text{iff } a \in f^{-1}(W) \text{ or } a \in f^{-1}(Z) \\ \text{iff } a \in f^{-1}(W) \cup f^{-1}(Z) \\ \text{Thus, } f^{-1}(W \cup Z) = f^{-1}(W) \cup f^{-1}(Z) \end{array}}$

(3) Let's show that

$$A - f^{-1}(W) = f^{-1}(B - W)$$

We have that $a \in A - f^{-1}(W)$

iff $a \in A$ and $a \notin f^{-1}(W)$



iff $f(a) \in B$ and $f(a) \notin W$.

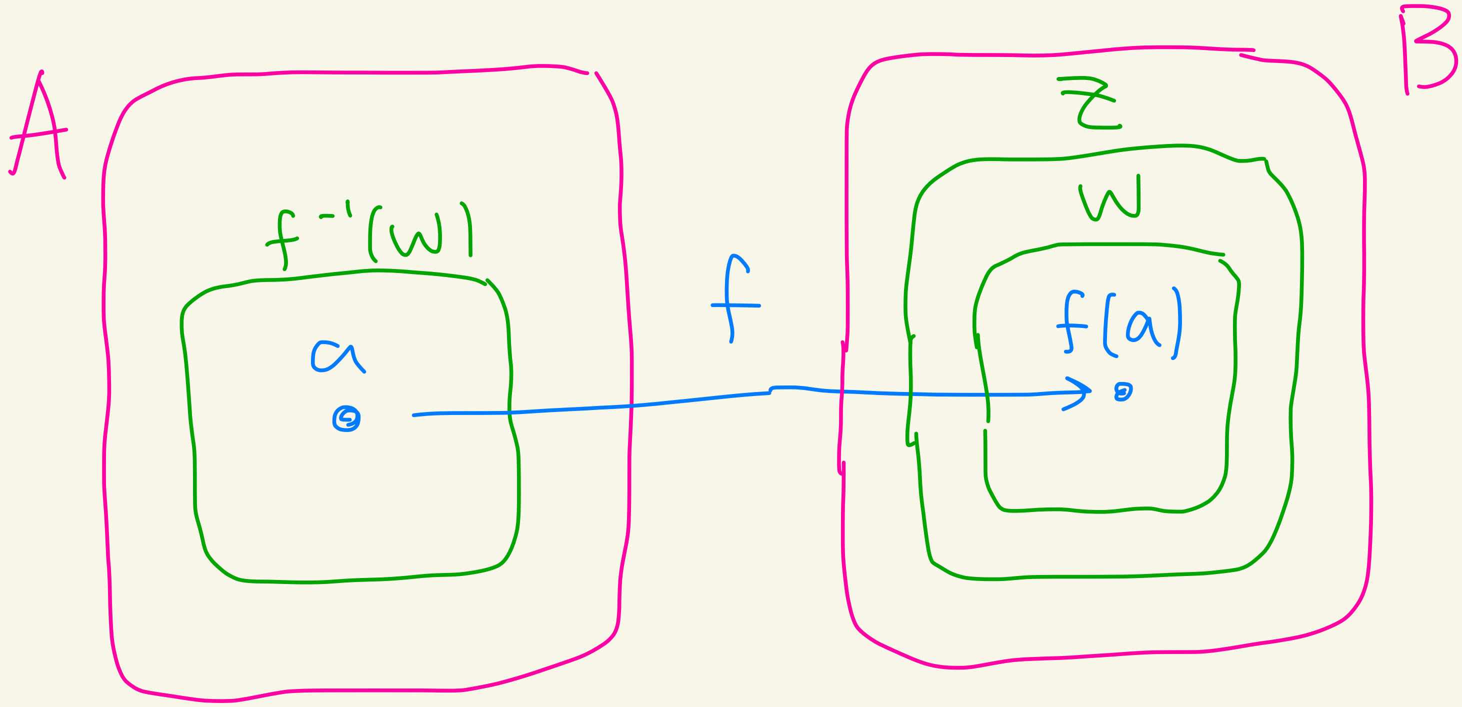
iff $f(a) \in B - W$.

Thus, $A - f^{-1}(W) = f^{-1}(B - W)$.

④ Suppose that $W \subseteq Z$.

Let's prove that $f^{-1}(W) \subseteq f^{-1}(Z)$.

Let $a \in f^{-1}(W)$.



Then, $f(a) \in W$.

Since $f(a) \in W$ and $W \subseteq Z$, we know that $f(a) \in Z$.

Thus, $a \in f^{-1}(Z)$.

Hence, $f^{-1}(W) \subseteq f^{-1}(Z)$.